

ECE 438 Homework 11, due in class Friday, 11/12/2004.

Problem 1. Suppose A_1 and A_2 are two random variables with the following joint PDF:

$$f_{A_1, A_2}(a_1, a_2) = \begin{cases} \frac{1}{2}, & \text{if both } 0 \leq a_1 \leq 1 \text{ and } 0 \leq a_2 \leq 1; \\ \frac{1}{2}, & \text{if both } 1 \leq a_1 \leq 2 \text{ and } 1 \leq a_2 \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the marginal PDF of A_1 , $f_{A_1}(a_1)$.
- (b) (ONE-BIT MAX QUANTIZER.) Suppose we want to quantize A_1 to one bit. In other words, we have two quantization intervals, $[x_1, x_2)$ and $[x_2, x_3]$, and two quantization levels, q_1 and q_2 : any value within $[x_1, x_2)$ gets quantized to q_1 ; any value within $[x_2, x_3]$ gets quantized to q_2 . Find x_1, x_2, x_3, q_1 , and q_2 which minimize the mean square error,

$$E[(\hat{A}_1 - A_1)^2], \tag{1}$$

where \hat{A}_1 is the result of quantizing A_1 . (The reason that this is called a one-bit quantizer is that we need just one bit to represent the result of quantization: there are only two quantization levels.)

- (c) Calculate the mean-square error (Eq. (1)) for the quantizer that you obtained in Part (b).
- (d) Show that the marginal PDF of A_2 is the same as the marginal PDF of A_1 . Show that therefore, the one-bit Max quantizer for A_2 is identical to the one for A_1 which you obtained in Part (b), and that

$$E[(\hat{A}_2 - A_2)^2] = E[(\hat{A}_1 - A_1)^2]. \tag{2}$$

- (e) Note that, in the above procedure, we used two bits to quantize A_1 and A_2 (one bit for A_1 and one bit for A_2). Consider an alternative procedure for quantizing the same variables, using the same number of bits:

$$\begin{aligned} \text{If } A_1 + A_2 < 1, & \text{ set } \hat{A}'_1 = \frac{1}{3}; \\ \text{if } 1 \leq A_1 + A_2 < 2, & \text{ set } \hat{A}'_1 = \frac{2}{3}; \\ \text{if } 2 \leq A_1 + A_2 < 3, & \text{ set } \hat{A}'_1 = \frac{4}{3}; \\ \text{if } A_1 + A_2 \geq 3, & \text{ set } \hat{A}'_1 = \frac{5}{3}; \\ \text{in every case, set } & \hat{A}'_2 = \hat{A}'_1. \end{aligned}$$

In other words, the input to this new quantizer is the pair (A_1, A_2) , and the output is the pair of the quantized values (\hat{A}'_1, \hat{A}'_2) . (A quantization strategy where two or more variables are quantized jointly is called *vector quantization*). This particular quantizer can be viewed as using both bits to quantize A_1 (since there are four quantization levels for A_1), and using zero bits to quantize A_2 : once \hat{A}'_1 is stored, \hat{A}'_2 does not need to be stored at all, since it is equal to \hat{A}'_1 .

For this new quantizer, find the mean-square errors,

$$E[(\hat{A}'_1 - A_1)^2] \text{ and } E[(\hat{A}'_2 - A_2)^2]. \quad (3)$$

Compare your results with Eq. (1) and (2), and explain why this new quantizer is able to achieve smaller errors with the same number of bits.

(Hint. For Problem 1, it may be very helpful to draw, in the a_1 - a_2 plane, the set of all points where the joint PDF of A_1 and A_2 is non-zero. In the same picture, draw the quantization points for the two quantizers.)

Problem 2. Suppose that an analog speech signal has a bandwidth of 4 kHz, and can be modeled as a stationary random process with a marginal probability density that is Gaussian with mean zero and variance 1. The goal of this exercise is to design a system to sample, quantize, and store the speech signal on a computer.

- (a) Find the Nyquist sampling rate for the data.
- (b) Choose a quantizer range $[-A, A]$ so that the probability of overload (i.e., the probability that a sample of the speech signal is outside of this range) is approximately 0.04. **(Hint.** You will need to use a table for the cumulative distribution function of a Gaussian random variable.)
- (c) Let $X(n)$ and $Y(n)$ be the speech signal before and after quantization, respectively, and let $D(n) = Y(n) - X(n)$ be the quantization error. Assuming the quantization error $D(n)$ may be modeled as a random variable which is uniformly distributed between $-\Delta/2$ and $\Delta/2$, express the average signal-to-noise power ratio in terms of Δ . The average signal-to-noise power ratio is defined as follows:

$$\text{SNR} = 10 \log_{10} \frac{E[(X(n))^2]}{E[(D(n))^2]}$$

- (d) Assume that a B -bit uniform quantizer is used; also assume that the maximum quantization level is $A - \Delta/2$, and the minimum quantization level is $-A + \Delta/2$, where A is as found in Part (b). Find the minimum number of bits B so that $\text{SNR} \geq 40$.
- (e) Using the results of (d) and (a), find the number of bits per second required to digitize the speech data.