ECE 438
Homework 3, due in class Friday, 9/10/2004.

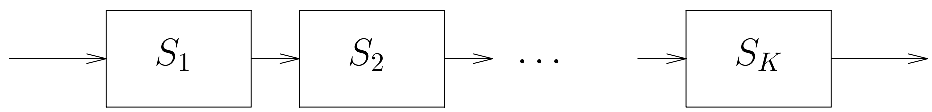

Figure 1: System S for Problem 1.

Problem 1. Suppose system S is a series connection of several systems $S_1, S_2, \ldots, S_K$, as shown in Fig. 1. In other words,

$$S[x] = S_K[S_{K-1}[\ldots S_2[S_1[x]]\ldots]].$$

(a) If each of the systems $S_1, \ldots, S_K$ is BIBO stable, can the overall system S be BIBO unstable?

(b) If S is BIBO stable, can one of the systems $S_1, \ldots, S_K$ be BIBO unstable?

Problem 2. Prove the following properties of the inner product and the norm in $\mathbb{C}^N$:

1. $\langle \mathbf{g}, \mathbf{s}\rangle = \langle \mathbf{s}, \mathbf{g}\rangle^*$.
2. $\langle a_1\mathbf{s}_1 + a_2\mathbf{s}_2, \mathbf{g}\rangle = a_1\langle \mathbf{s}_1, \mathbf{g}\rangle + a_2\langle \mathbf{s}_2, \mathbf{g}\rangle$.
3. $\langle \mathbf{s}, a_1\mathbf{g}_1 + a_2\mathbf{g}_2\rangle = a_1^*\langle \mathbf{s}, \mathbf{g}_1\rangle + a_2^*\langle \mathbf{s}, \mathbf{g}_2\rangle$.
4. $\|a\mathbf{s}\| = |a| \cdot \|\mathbf{s}\|$.
5. Pythagoras's theorem: the sum of energies of two orthogonal vectors is equal to the energy of their sum, i.e.,

$$\text{if } \langle \mathbf{s}, \mathbf{g}\rangle = 0, \text{ then } \|\mathbf{s}\|^2 + \|\mathbf{g}\|^2 = \|\mathbf{s}+\mathbf{g}\|^2.$$

Problem 3. Geometry of the Space $\mathbb{C}^N$ of N-point Signals.

(a) Show that the length of the projection $\mathbf{s_g}$ of a vector $\mathbf{s}$ onto another vector $\mathbf{g}$ does not exceed the length of $\mathbf{s}$:

$$\|\mathbf{s_g}\| \le \|\mathbf{s}\|. \tag{1}$$

Note that this result generalizes the following 2-D picture:

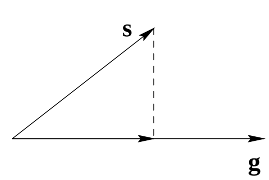

Figure 2: A vector cannot be shorter than its orthogonal projection onto another vector.

(**Hint.** Use the fact that $\mathbf{s}-\mathbf{s_g}$ is orthogonal to $\mathbf{s_g}$. This means that you can apply Pythagoras's theorem to the right triangle whose sides are $\mathbf{s}-\mathbf{s_g}$ and $\mathbf{s_g}$, and whose hypotenuse is $\mathbf{s}$.)

(b) Prove the Cauchy-Schwartz inequality:

$$|\langle \mathbf{g}, \mathbf{s} \rangle| \leq \|\mathbf{g}\| \|\mathbf{s}\|$$

(**Hint.** Use inequality (1), proved in Part (a). In (1), write $\mathbf{s}_\mathbf{g}$ in terms of $\mathbf{s}$ and $\mathbf{g}$. Then use the property $\|a\mathbf{g}\| = |a|\|g\|$ and cancel some terms.)

(c) Prove the triangle inequality: the sum of the lengths of two sides of a triangle cannot be smaller than the length of its third side, i.e.,

$$\|\mathbf{s}\| + \|\mathbf{g}\| \geq \|\mathbf{s} + \mathbf{g}\|$$

Note that this result generalizes the following 2-D picture:

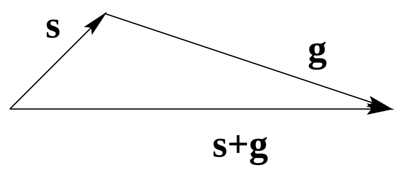

Figure 3: A side of a triangle cannot be longer than the sum of the lengths of the two other sides, i.e., $\|\mathbf{s} + \mathbf{g}\| \leq \|\mathbf{s}\| + \|\mathbf{g}\|$.

(**Hint.** Square both sides of the inequality. Then use the fact that $\|\mathbf{s}+\mathbf{g}\|^2 = \langle \mathbf{s}+\mathbf{g}, \mathbf{s}+\mathbf{g} \rangle$, and write this out. Then think how to use the Cauchy-Schwartz inequality proved in Part (b).)

(d) Prove that, for any real numbers $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$, the following inequality holds:

$$\sum_{n=1}^{N} (a_n + b_n)^2 \leq \left[\sqrt{\sum_{n=1}^{N} a_n^2} + \sqrt{\sum_{n=1}^{N} b_n^2} \right]^2$$

Problem 4. Complex Exponentials as Eigenfunctions of LTI Systems.
For each of the two discrete-time systems S_1, S_2 below, the indicated input/output pair represents the result of one experiment with the corresponding system. Decide whether the output $y(n)$ and input $x(n)$ of this system definitely cannot, possibly could, or must satisfy a convolution relationship of the form $y(n) = x * h(n)$ for some appropriate impulse response $h(n)$. Choose the statement that applies, and explain your reasoning. If your answer is "possibly could" or "must", determine an impulse response $h(n)$ and a frequency response $H(e^{j\omega})$ that would account for the given input/output pair.

(i) The response of system S_1 to input $e^{j(\pi/5)n}$ is $e^{j(\pi/7)n}$.

(ii) The response of system S_2 to input $e^{j(\pi/6)n}$ is $[4e^{j(\pi/12)n}]^2$.

Problem 5. Discrete-Time Fourier Series.
Consider the following DT complex exponential functions:

$$s_k(n) = \exp\left(\frac{j2\pi kn}{N}\right), \quad n = 0, \ldots, N-1; \quad k = 0, \ldots, N-1. \tag{2}$$

In other words, there are N functions, $s_0(n), s_1(n), \ldots, s_{N-1}(n)$, and each of them is defined for $n = 0, 1, \ldots, N-1$. As usual, we are identifying each of these functions with a vector:

$$\mathbf{s}_k = \left(\exp\left(\frac{j2\pi k0}{N}\right),\ \exp\left(\frac{j2\pi k1}{N}\right), \ldots, \exp\left(\frac{j2\pi k(N-1)}{N}\right) \right)^T.$$

(a) Show that these functions are pairwise orthogonal. In other words, show that

$$\text{if } k \neq m, \text{ then } \langle \mathbf{s}_k, \mathbf{s}_m \rangle = 0, \text{ for any } N.$$

(b) Find the energy of each of these complex exponentials, i.e. find $\|\mathbf{s}_k\|^2$ for $k = 0, 1, \ldots, N-1$.

(c) Suppose that $N = 438$. Let function $s(n)$ be defined by:

$$s(n) = 1 + \cos(\pi n/3) + j\sin(\pi n/3), \quad n = 0, \ldots, 437.$$

Compute the coefficients $a_0, a_1, \ldots, a_{437}$ in the Fourier series expansion:

$$\mathbf{s} = \sum_{k=0}^{437} a_k \mathbf{s}_k,$$

where $\mathbf{s} = (s(0), s(1), \ldots, s(437))^T$, and $\mathbf{s}_k$ are the complex exponentials defined above. (**Hint**. You do not have to compute any inner products here. Try to write the signal as a linear combination of complex exponentials and find the coefficients by inspection.)

Problem 6. GEOMETRIC INTERPRETATION OF FOURIER SERIES.
The following system S takes 2-point real-valued signals as inputs, and produces 2-point signals as outputs. The system is specified by the following input-output relationship:

$$y(n) = \frac{1}{\sqrt{2}} \sum_{k=1}^{2} x(k) \exp\left(j\pi(k-1)(2-n)\right), \text{ for } n = 1, 2,$$

where $\mathbf{y} = S[\mathbf{x}] = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix}$ is the response of the system to the input $\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in \mathbb{R}^2$.

(a) Show that S is a rotation of the plane $\mathbb{R}^2$. In other words, show that, for any vector $\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in \mathbb{R}^2$, the response of the system $\mathbf{y} = S[\mathbf{x}]$ is the rotation of $\mathbf{x}$ by a certain angle θ (see Fig. 4), where θ is the same for all inputs $\mathbf{x}$. Find θ.

Hints. One method would be to write $\mathbf{x}$ in polar coordinates: $x(1) = R\cos\phi$, $x(2) = R\sin\phi$, and consider the vector $\mathbf{x}_\theta$, the result of rotating $\mathbf{x}$ counterclockwise by angle θ:

$$x_\theta(1) = R\cos(\phi + \theta), \quad (3)$$
$$x_\theta(2) = R\sin(\phi + \theta). \quad (4)$$

Now use trigonometric identities (such as $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$) to massage the two expressions (3) and (4), and write $\mathbf{x}_\theta$ in terms of $\mathbf{x}$. You should get an expression of the form:

$$\mathbf{x}_\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathbf{x}, \quad (5)$$

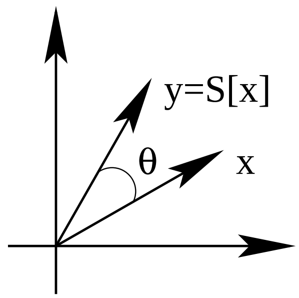

Figure 4: Illustration to Problem 2: in 2-D, the system S rotates the input by some angle θ.

where the coefficients A, B, C, and D depend on θ. Go back to the definition of the system S, and write it in the same form:

$$\mathbf{y} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \mathbf{x}, \tag{6}$$

where A_1, B_1, C_1, and D_1 are numbers. Now find θ by identifying coefficients in Eq. (5) and Eq. (6), and prove that Eq. (6) indeed describes a rotation. It may be helpful to draw a few pictures in the plane before and during your calculations.

(b) Let $\mathbf{x}$ and $\mathbf{v}$ be two real-valued 2-point input signals, such that the distance between them is 7:

$$\sqrt{(x(1) - v(1))^2 + (x(2) - v(2))^2} = 7.$$

Let $\mathbf{y}$ and $\mathbf{w}$ be the responses of S to these signals, respectively: $\mathbf{y} = S[\mathbf{x}]$, and $\mathbf{w} = S[\mathbf{v}]$. Calculate the following quantity:

$$\sqrt{(y(1) - w(1))^2 + (y(2) - w(2))^2}.$$

Suppose that the angle between $\mathbf{x}$ and $\mathbf{v}$ is $5\pi/8$. What is the angle between $\mathbf{y}$ and $\mathbf{w}$? For all parts, **justify your answers**.

Problem 7. Suppose that $x(n) = e^{j(\pi/4)n}$ and $h(n) = \delta(n) + \delta(n-1)$. Does there exist a number A such that the following equation is satisfied for all n?

$$x * h(n) = Ax(n).$$

If so, find this number.