ECE 438.
Homework 4, due in class Friday, 9/17/2004.

Problem 1. DTFT PROPERTIES.
Prove the following DTFT properties:

(a) For any signal x, the DTFT of $nx(n)$ is $j\frac{dX(e^{j\omega})}{d\omega}$. (**Hint**. Use the inverse DTFT formula and integrate by parts.)

(b) For any signals x and y, the DTFT of $x(n)y(n)$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta.$$

(**Hint**. Use the inverse DTFT formula and change the order of the two integrations.)

Problem 2. DTFT PAIRS.

(a) If a is a number such that $|a| < 1$, show that the DTFT of $a^n u(n)$ is

$$\frac{1}{1 - ae^{-j\omega}}.$$

(b) Using the results from Part (a) and Problem 1(a), find the DTFT of $na^n u(n)$.

Problem 3. DFT PROPERTIES.
Derive the following properties of the N-point DFT (assume that all the signals and DFT's below are periodic with period N).

1. Delay:
$$x(n - n_0) \leftrightarrow X(k)e^{-j\frac{2\pi n_0}{N}k}.$$

2. Modulation:
$$x(n)e^{j\frac{2\pi k_0}{N}n} \leftrightarrow X(k - k_0).$$

3. Reciprocity:
$$X(n) \leftrightarrow Nx(-k).$$
I.e., to get the DFT of the DFT of x, flip x around the origin and multiply by N.
(**Hint.** Use the inverse DFT formula to express $x(k)$ in terms of $X(n)$, then show that $Nx(-k)$ is the DFT of $X(n)$.)

4. Parseval's theorem:
$$\mathbf{x}^H\mathbf{y} = \frac{1}{N}\mathbf{X}^H\mathbf{Y},$$

where, as usual, $\mathbf{x}$ and $\mathbf{y}$ represent signals $x(n)$ and $y(n)$ as column vectors:

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{pmatrix},$$

and $\mathbf{X}$, $\mathbf{Y}$ are their respective DFT's:

$$\mathbf{X} = \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{pmatrix}.$$

$\mathbf{x}^H$ and $\mathbf{X}^H$ denote conjugate transposes of $\mathbf{x}$ and $\mathbf{X}$, respectively:

$$\mathbf{x}^H = \begin{pmatrix} x^*(0) & x^*(1) & \dots & x^*(N-1) \end{pmatrix},$$
$$\mathbf{X}^H = \begin{pmatrix} X^*(0) & X^*(1) & \dots & X^*(N-1) \end{pmatrix}.$$

(**Hint.** Use the fact that

$$\mathbf{x}^H\mathbf{y} = (B\mathbf{X})^H(B\mathbf{Y}),$$

where B is the inverse DFT matrix described in class. Then use the rules for transposing a matrix product, as well as a property of B derived in class, to simplify the right-hand side.)

5. The relationship between DFT and DTFT.
Let $x_z(n)$ be a non-periodic signal defined as follows:

$$x_z(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise}, \end{cases}$$

and let $X_z(e^{j\omega})$ be the DTFT of $x_z(n)$. Show that the DFT of $x(n)$ is a discretization of the DTFT of $x_z(n)$:

$$X(k) = X_z(e^{j\frac{2\pi k}{N}}).$$

Problem 4. DFT, Upsampling, and Replication.
Let $X(k)$ be the N-point DFT of the sequence $x(n)$, $0 \le n \le N-1$.

(a) We define a $2N$-point sequence $y(n)$ as follows:

$$y(n) = \begin{cases} x\left(\frac{n}{2}\right), & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

Express the $2N$-point DFT of $y(n)$ in terms of $X(k)$. (Remember that $X(k)$ is an N-point sequence.)

(b) We define a $2N$-point sequence $w(n)$ as follows:

$$w(n) = \begin{cases} x(n), & \text{if } n = 0, 1, \dots, N-1, \\ x(n-N), & \text{if } n = N, N+1, \dots, 2N-1. \end{cases}$$

Express the $2N$-point DFT of $w(n)$ in terms of $X(k)$. (Remember that $X(k)$ is an N-point sequence.)

Problem 5. Discrete Cosine Transform.
Consider the following N-point signals defined for $n = 0, \ldots, N-1$:

$$
\begin{aligned}
g_0(n) &= \sqrt{\frac{1}{N}}, \\
g_k(n) &= \sqrt{\frac{2}{N}} \cos\left(\frac{k\pi}{N}\left(n + \frac{1}{2}\right)\right), \quad \text{for } k = 1, \ldots, N-1.
\end{aligned}
$$

As usual, we identify each signal $g_k(n)$ with the following N-dimensional vector $\mathbf{g}_k \in \mathbb{C}^N$:

$$
\mathbf{g}_k = (g_k(0)\ g_k(1)\ \ldots\ g_k(N-1))^T, \quad \text{for } k = 0, \ldots, N-1.
$$

(a) Show that these signals are pairwise orthogonal. In other words, show that

$$
\text{if } k \neq m, \text{ then } \langle \mathbf{g}_k, \mathbf{g}_m \rangle = 0, \text{ for any } N.
$$

Hints.
1. Consider the cases $\langle \mathbf{g}_0, \mathbf{g}_m \rangle$ and $\langle \mathbf{g}_k, \mathbf{g}_m \rangle$ for $k \neq 0$ separately.
2. Use the identity $\cos\alpha = (\exp(j\alpha) + \exp(-j\alpha))/2$ to write the inner product as the summation of complex exponentials. Then use a strategy similar to Problem 5 from Homework 3: apply the geometric series formula and simplify.

(b) Find the energy of each of these signals, i.e. find $\|\mathbf{g}_k\|^2$ for $k = 0, 1, \ldots, N-1$.

(c) Conclude from Part (a) that the N vectors $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{N-1}$ form an orthogonal basis for $\mathbb{C}^N$. (This basis is called a discrete cosine basis.) Use this fact to find formulas for the coefficients $a_1, \ldots, a_{N-1}$ in the following expansion of an N-point signal $s(n)$:

$$
s(n) = \frac{a_0}{\sqrt{N}} + \sum_{k=1}^{N-1} \sqrt{\frac{2}{N}} a_k \cos\left(\frac{k\pi}{N}\left(n + \frac{1}{2}\right)\right).
$$

Your formulas must express each coefficient a_k in terms of the samples of the signal $s(n)$. These coefficients are called the discrete cosine transform (DCT) of the signal $s(n)$. A somewhat different variety of DCT is at the heart of JPEG image compression standard (you use it any time you open or save a picture file with extension .jpg). JPEG partitions an image into 8×8 blocks, computes the DCT coefficients for every block, and encodes the coefficients.

(d) Let

$$
s(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right).
$$

Let $N = 2004$. Find the DCT coefficients $a_0, \ldots, a_{2003}$ in the following expansion:

$$
s(n) = \frac{a_0}{\sqrt{2004}} + \sum_{k=1}^{2003} \frac{a_k}{\sqrt{1002}} \cos\left(\frac{k\pi}{2004}\left(n + \frac{1}{2}\right)\right).
$$