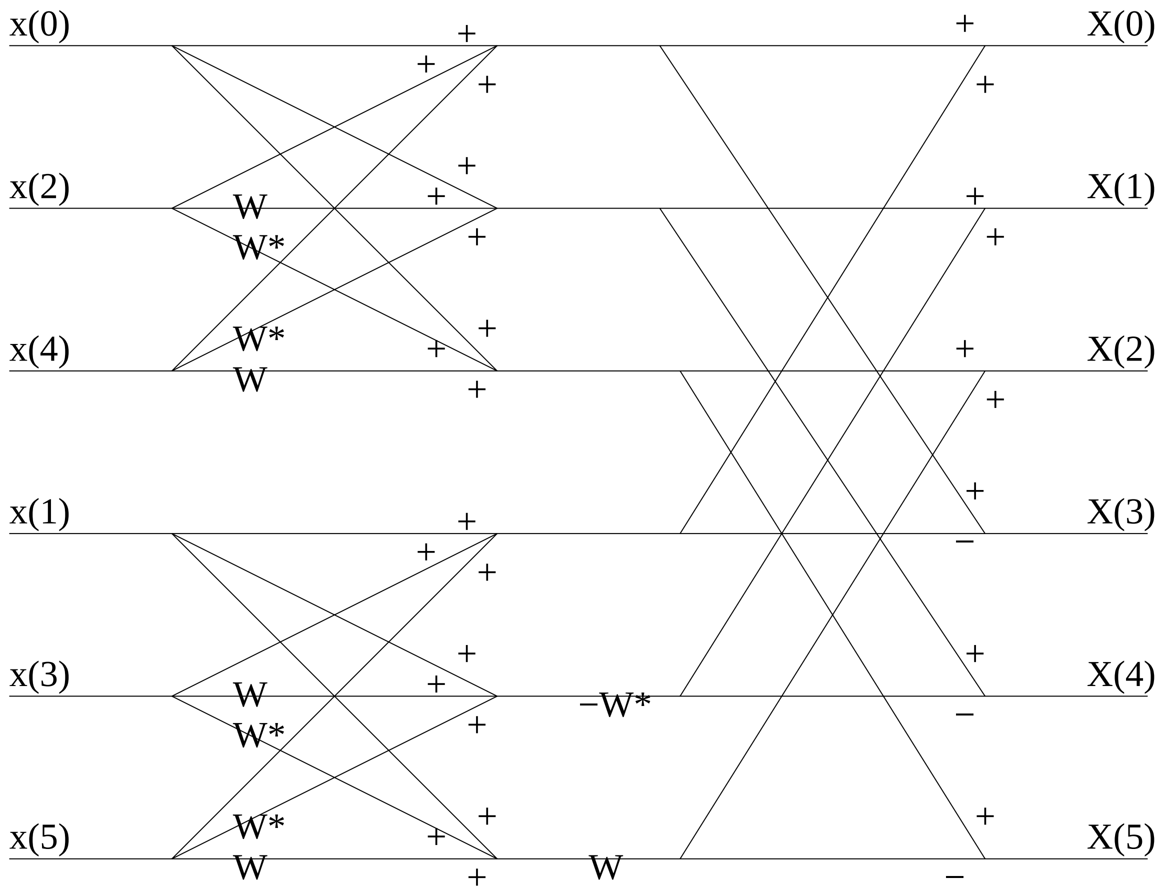


Figure 1: Diagram for Problem 5. $W = -0.5 - j0.5\sqrt{3}$.

ECE 438.
Homework 5, due in class Friday, 9/24/2004.

Problem 1. Fast Fourier Transform.
Consider the algorithm whose diagram is in Fig. 1. In the diagram, $W = -0.5 - j0.5\sqrt{3}$, and W^* is the complex conjugate of W. Multiplications by W, W^*, and $-W^*$ are denoted by putting these numbers on the corresponding links in the diagram. For example, the expression for $X(1)$ specified by the diagram is:

$$X(1) = [x(0) + Wx(2) + W^*x(4)] + (-W^*)[x(1) + Wx(3) + W^*x(5)].$$

Does this algorithm compute the 6-point DFT of the signal $x(n)$? Fully substantiate your answer. (E.g., if your answer is yes, you need to show that the $X(k)$'s on the right-hand side of the diagram are indeed the DFT of the signal whose samples $x(n)$ are on the left-hand side of the diagram.) What is the total number of complex multiplications required by this algorithm? (Do not count multiplications by 1 or -1.) What is the total number of complex multiplications for computing a six-point DFT using the definition, $X(k) = \sum x(n)e^{-j2\pi kn/6}$?

Problem 2. Convolution and periodic convolution.
(a) Let x and h be two signals of duration N, with $x(n) = h(n) = 0$ for $n < 0$ and for $n > N-1$. Let x_p and h_p be periodic extensions of x and h, respectively, with period N–in other words, let x_p and h_p be N-periodic signals such that

$$x_p(n) = x(n) \text{ and } h_p(n) = h(n) \text{ for } n = 0, 1, \ldots, N-1.$$

Let $y(n) = x * h(n)$ be the convolution of x and h.

(a) Let $y_p(n) = x \circledast h(n)$ be the N-point periodic convolution of x and h, defined by:

$$x \circledast h(n) = \sum_{m=0}^{N-1} x_p(m) h_p(n-m).$$

Prove that the following relationship holds:

$$y_p(n) = \begin{cases} y(n) + y(n+N), & n = 0, 1, \ldots, N-2 \\ y(N-1), & n = N-1. \end{cases}$$

(Note that, as usual, your proof must be general. Illustrating this property for some specific x and h does not constitute a proof.)

(b) How can the convolution y of the signals x and h be calculated using DFT? (Note that multiplying the N-point DFT's of x and h and taking the inverse DFT will produce $y_p(n)$, not $y(n)$!)

Problem 3. Periodic convolution.
Consider the following DT signals:

$$\begin{aligned} x(n) &= \delta(n) + 2\delta(n-1) + \delta(n-2) + \delta(n-3) \\ h(n) &= \delta(n) + 2\delta(n-1) + 3\delta(n-2) \end{aligned}$$

Compute and plot the N-point periodic convolution of $x(n)$ and $h(n)$ for:

(a) $N = 4$;

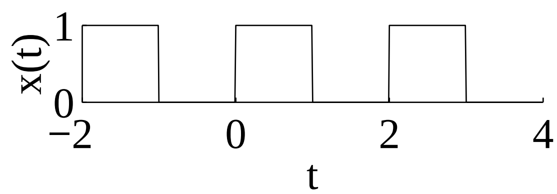
(b) $N = 5$;

(c) $N = 100$.

Do not use Matlab or any other electronic means.

Problem 4. Reconstruction of a continuous-time signal from its samples.
Consider one of the periodic signals for which you calculated the continuous-time Fourier series expansion in Lab 3, Section 2.1:

$$x(t) = \begin{cases} 1, & \text{if } 2n \le t < 2n+1, \quad \text{for any integer } n, \\ 0, & \text{if } 2n-1 \le t < 2n, \quad \text{for any integer } n. \end{cases}$$



Suppose we use the ideal sampling model considered in class, to sample and reconstruct this signal. First, in order to avoid aliasing, we filter the signal using an ideal analog low-pass filter $H'(f)$ with cut-off frequency $f_s/2$:

$$H'(f) = \begin{cases} 1, & |f| \leq \frac{f_s}{2} \\ 0, & \text{otherwise.} \end{cases}$$

We then sample the filtered signal at a rate of f_s samples per second, using an ideal sampler (i.e. multiplication by a periodic train of ideal continuous-time impulses).

From the continuous-time impulse train obtained through sampling, we reconstruct a signal $y(t)$ using another ideal analog low-pass filter, $H(f)$:

$$H(f) = \begin{cases} \frac{1}{f_s}, & |f| \leq \frac{f_s}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Find the reconstructed signal $y(t)$ for the following values of f_s.

(i) 0.5 Hz.

(ii) 1.5 Hz.

(iii) 2.5 Hz.

(iv) 10.5 Hz.

(For Parts (i), (ii), (iii), give expressions for $y(t)$ without using MATLAB. For Part (iv), use MATLAB to plot $y(t)$.) Comment on your results.

Problem 5. ALIASING.
The continuous-time signal $x_c(t) = 3\cos(400\pi t) + 5\sin(1200\pi t) + 6\cos(4400\pi t) + 2\sin(5200\pi t)$ is ideally sampled at the rate of 4 kHz generating a discrete-time signal $x(n) = A_1\cos(\omega_1 n) + A_2\sin(\omega_2 n) + A_3\cos(\omega_3 n) + A_4\sin(\omega_4 n)$, where $0 \leq \omega_1 \leq \pi$, $0 \leq \omega_2 \leq \pi$, $0 \leq \omega_3 \leq \pi$, $0 \leq \omega_4 \leq \pi$. Find the numbers A_1, A_2, A_3, A_4, ω_1, ω_2, ω_3, and ω_4.