ECE 438 Homework 8, due in class Friday, 10/22/2004.

Problem 1. Consider the following joint PDF of two continuous random variables, X and Y:

$$f_{XY}(x,y) = \begin{cases} A, & 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find A.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y uncorrelated? Are they statistically independent?

(d) Find $E[Y]$ and $Var(Y)$.

(e) Let $Z = X + Y$. Find the conditional PDF $f_{Z|X}(z|x)$, as well as the PDF $f_Z(z)$ of Z.

Problem 2. Let X be a continuous random variable, uniformly distributed between -1 and 1, i.e.,
$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

(a) Find $E[X]$.

(b) Find $var(X)$.

(c) Find the PDF of the random variable $Y = X + 1$.

(d) Find the correlation coefficient of $Y = X + 1$ and X.

(e) Find the CDF of X.

(f) Find the CDF of $Z = 2X$ and differentiate to find the PDF of Z.

Problem 3. A signal $s = 3$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$. When the weather is good, which happens with probability 2/3, W is a Gaussian random variable with zero mean and variance 4. When the weather is bad, W is normal with zero mean and variance 9. In the absence of any weather information:

(a) What is the PDF of X?

(b) Calculate the probability that X is between 2 and 4.

A table of values of the standard Gaussian CDF is available here:
http://www.ece.purdue.edu/~ipollak/ee438/FALL04/homeworks/Gaussian_table.pdf

Problem 4. Variable X_{st}, the *standardized* random variable for random variable X, is given by

$$X_{st} = \frac{X - m_X}{\sigma_X},$$

where $m_X = E[X]$ is the expectation of X, and σ_X is the standard deviation of X, $\sigma_X = \sqrt{Var(X)}$.

(a) Determine $E[X_{st}]$ and $Var(X_{st})$.

(b) Recall that we defined the correlation coefficient as follows:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

Show that $\rho_{XY} = E[X_{st}Y_{st}]$. Determine the numerical value of ρ_{XY} if

(i) $X = aY$, $a > 0$.

(ii) $X = -aY$, $a > 0$.

(iii) X and Y are statistically independent.

(iv) $X = aY + b$, $a > 0$.

(c) For each performance of the experiment, the experimental value of random variable Y_{st} is to be approximated by cX_{st}. Prove that the value of constant c which minimizes the expected *mean square error*, $E[(Y_{st} - cX_{st})^2]$, for this approximation is given by $c = \rho_{XY}$.