Stiffness Matrix and Quantitative Measure of Formation Rigidity

Guangwei Zhu, Jianghai Hu

***Abstract*— Rigidity of formation is an importance concept in multi-agent localization and control problems. There are well-developed existing methods to test the rigidity of a given graph. However, little work is done on quantitative measurement of formation rigidity. In this paper, we propose the stiffness matrix of a fomation representing both its rigidity and structural information, from which we then derive the worst-case rigidity index, as an applicable quantitative measure of formation rigidity. Its validity is shown through the illustration of its related properties, which conform to intuitive assumptions and practical applications.**

I. INTRODUCTION

Formation control of multi-agent systems consisting of robots or unmanned vechicles with sensors and actuators has been an ongoing topic in the research field. In many applications, such as multi-agent antenna arrays, persisting a stable formation is crucial to the functionality and efficiency of the entire multi-agent network. Lots of research work has been done from the perspective of control strategy for the system as well as the persistence of the formation itself. Rigidity theory then becomes growingly popular in the research on the latter topic.

Much work has been done to determine whether a formation, or more generally speaking, a graph is (globally) rigid. A formation underlaid by a rigid graph usually benefits from its deterministic and unique realization. This characteristic is often of considerable importance in multi-agent system, as it often leads to localizability of agents and stability of the entire formation. The rigidity test is mainly based on two fundamental theorems, the rigidity matrix theorem [1], [2] and Laman's theorem [3]. These theorems have been widely used in the field of multi-agent system [4], examples of which include stabilization of multi-vehicle systems [5], [6], and localization of wireless sensor networks [7], [8].

Although mature theory has been developed to test the rigidity of graphs, little work is done on the comparison among rigid formations. Such comparison, however, has practical application in the analysis and design process of formation. For instance, in a wireless sensor network localization problem [8], one may choose from several formations for sensors to minimize the uncertainty of their locations after localization process under measurement error. Another example is formation control of multi-vehicle system, where it is best to implement a formation that is "easy" to persist, in the sense that the controllers are more responsive to the perturbation acting upon the system. These applications lead to the need of a quantitative measure of formation rigidity, which can provide instructive information in formation design and evaluation.

This paper contributes to the above problem by proposing two new quantities for a graph, the *stiffness matrix* and the *worst-case rigidity index*. We first derive the stiffness matrix from a basic spring-mass system analogy, and then show some of its properties and the underlying relation to rigidity matrix in classic rigidity theory. Then, we introduce the worst-case rigidity index, which is a scalar value derived from the stiffness matrix. Several properties of this index are studied to show its validity and practical value as a measure of formation rigidity. Finally, we give some examples to illustrate how the worst-case rigidity index can be used to evaluate formations comparatively.

In Section II, we define some notations that are used throughout this paper. We introduce the concept of stiffness matrix of a formation in Section III and its properties in Section IV. Then, in Section V, we derive the worst-case rigidity index, and demonstrate its validity as a quantitative measure of formation rigidity. We give some examples to illustrate the application of the worst-case rigidity index in Section VI. We conclude in Section VII with some perspective work related to the quatitative measurement of formation rigidity.

II. NOTATION

In this section, the notation conventions used in this paper is introduced.

$\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}$ denotes the set of real numbers, n-dimensional real column vectors, and n-by-m matrices, respectively. $\mathbb{R}^+$ denotes the set of positive real numbers. Italic lowercase letters, with or without subscripts, e.g., k, p_{ij}, represent scalar variables or constants. Italic uppercase letters, such as R, K, S_{ij}, are matrices. The tranpose of A is denoted as $A^\top$. Calligraphic letters, e.g., $\mathcal{I}, \mathcal{C}$, denote general sets. Column cectors are denoted by bold letters. A vector is called a *multi-quantity* if it is a stacked vector composed by several quantities which themselves are vectors. A multi-quantity is denoted by a bold lowercase letter with no subscript, and each of its components is denoted by the same letter with a subscript. For example, $\mathbf{p} \in \mathbb{R}^{2N}$ is a multi-point, $\mathbf{p} = [\mathbf{p}_1^\top \quad \mathbf{p}_2^\top \quad \cdots \quad \mathbf{p}_N^\top]^\top$; each component $\mathbf{p}_i \in \mathbb{R}^2$ represents the two-dimensional coordinate of a point in a N-point system.

The inner product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined as $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^\top \mathbf{v} \in \mathbb{R}$. In this paper, the norm of a vector $\mathbf{v}$ particularly refers to its Euclidean norm, i.e., $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$. For matrix A, B, we write $A > 0$ if A is positive definite and $A \geq 0$ is A is non-negative definite, $A > B$ if $A - B > 0$ and $A \geq B$ if $A - B \geq 0$.

III. STIFFNESS MATRIX OF FORMATION

In this section, we propose the stiffness matrix of a formation to establish the linear relationship between a small perturbing force and resultant displacement on that formation.

A. Analogy of Spring-Mass System

Consider a simplistic formation where two agents try to maintain constant distance d along a one-dimensional space. Each agent is implemented with a P-controller and an actuator that outputs a force to move the agent. Let p_1, p_2 be the current positions of the two agents, $\hat{p}_1, \hat{p}_2$ be the coordinates of natural positions, and $\Delta p_1, \Delta p_2$ denote the small displacements, $\Delta p_i = p_i - \hat{p}_i$ $(i = 1, 2)$. Suppose $\hat{p}_2 - \hat{p}_1 = d$, we have

$$\begin{cases} \ddot{p}_1 = k_{21}(\Delta p_2 - \Delta p_1) \\ \ddot{p}_2 = k_{12}(\Delta p_1 - \Delta p_2) \end{cases} \tag{1}$$

with coefficients $k_{12}, k_{21} > 0$. If we suppose $k_{12} = k_{21}$, the senario is analogous to a spring-mass system, where two unit point masses are connected with a spring whose natural length is d and spring constant equals to k_{12}. Hence $\ddot{p}_i$ $(i = 1, 2)$ represents the force acting upon the i-th agent with unit mass. From (1) it is obvious that no force is acted upon the agents if $\Delta p_1 = \Delta p_2$, which describes a translation of the formation.

We now consider a general two-dimensional static formation of N agents. Let $\mathcal{I} = \{1, 2, \ldots, N\}$ be the set of indices of agents and a constant multi-point $\hat{\mathbf{p}} = [\hat{\mathbf{p}}_1^\top \ \ \hat{\mathbf{p}}_2^\top \ \ \cdots \ \ \hat{\mathbf{p}}_N^\top]^\top \in \mathbb{R}^{2N}$ be an initial configuration of the formation, each $\hat{\mathbf{p}}_i \in \mathbb{R}^2$ denoting the coordinate of the i-th agent. We assign a non-negative scalar k_{ij} to each pair of agents $(i, j) \in \mathcal{I}^2, i \neq j$, and $k_{ij} = k_{ji}$. This k_{ij} denotes the *connectivity coefficient* between the i-th and j-th agents. If $k_{ij} > 0$, then $\|\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j\| = d_{ij}$ is a distance constraint of the formation, while $k_{ij} = 0$ implies that $\|\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j\|$ is not constrained. Then we define the *connectivity matrix* $K = [k_{ij}] \in \mathbb{R}^{N \times N}$. We call $(\mathcal{I}, \hat{\mathbf{p}}, K)$ a *KP-formation*.

We can analogize a KP-formation to a mass-spring system. Each agent is represented by a mass point (for convenience we will call this a *point* in later description, and point i represents the i-th agent in the formation). If $k_{ij} > 0,\ i, j \in \mathcal{I}$, then we connect point i and j by a spring with natural length d_{ij} and spring constant k_{ij}. We may see that the formation is maintained when all the springs are at their natural length in the corresponding spring-mass system.

B. Perturbation Analysis of KP-Formation

The static formation is maintained when point i stays at their original location $\hat{\mathbf{p}}_i$ for each $i \in \mathcal{I}$. The locations may change if the system is subject to certain perturbation. We now analyze the quantitative relation between displacement and the causal perturbation.

Consider a KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$. We displace point i by a sufficiently small $\Delta \mathbf{p}_i$. The new coordinate of point i is now $\mathbf{p}_i$, $\mathbf{p}_i = \hat{\mathbf{p}}_i + \Delta \mathbf{p}_i$. It can be shown that

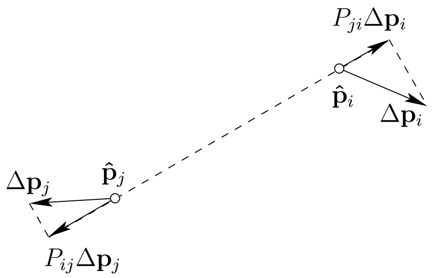

Fig. 1. Determining extended length of the spring using projection

$$\begin{aligned} |\|\mathbf{p}_i - \mathbf{p}_j\| - \|\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j\|| =& \|P_{ji}\Delta\mathbf{p}_i - P_{ij}\Delta\mathbf{p}_j\| \\ &+ o\left(\|\Delta\mathbf{p}_i - \Delta\mathbf{p}_j\|\right) \end{aligned} \tag{2}$$

where $P_{ij} \in \mathbb{R}^{2\times 2}$ is the projection matrix,

$$P_{ij} = \frac{1}{\|\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i}\|^2}(\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i})(\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i})^\top. \tag{3}$$

Let $\mathbf{e}_{ij} = (\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i})/\|\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i}\|$, then P_{ij} can be expressed as $P_{ij} = \mathbf{e}_{ij}\mathbf{e}_{ij}^\top$. It is obvious that P_{ij} and P_{ji} are equal.

Fig. 1 illustrates an intuitive approach towards this result. Notice that since both $\Delta\mathbf{p}_i$ and $\Delta\mathbf{p}_j$ are "small", change in the direction from point i to point j is negligible, and hence the projection matrices can be approximated using original coordinates of point i and j.

Recall that we have developed a spring-mass analogy for a KP-formation. If $k_{ij} > 0$, the displacement of point i and j may stretch the imaginary spring connecting them. By Hooke's Law, the elastic force acting upon point i is $k_{ij}\left(P_{ij}\Delta\mathbf{p}_j - P_{ji}\Delta\mathbf{p}_i\right)$, and $k_{ji}\left(P_{ji}\Delta\mathbf{p}_i - P_{ij}\Delta\mathbf{p}_j\right)$ upon point j. Then each point is acted upon a net force $\mathbf{f}_i$ where

$$\mathbf{f}_i = \sum_{j \in \mathcal{I}\setminus\{i\}} k_{ij}\left(P_{ij}\Delta\mathbf{p}_j - P_{ji}\Delta\mathbf{p}_i\right), \quad i \in \mathcal{I} \tag{4}$$

Let $\Delta\mathbf{p} = [\Delta\mathbf{p}_1^\top \ \ \Delta\mathbf{p}_2^\top \ \ \cdots \ \ \Delta\mathbf{p}_N^\top]^\top$ and $\mathbf{f} = [\mathbf{f}_1^\top \ \ \mathbf{f}_2^\top \ \ \cdots \ \ \mathbf{f}_N^\top]^\top \in \mathbb{R}^{2N}$. We call $\Delta\mathbf{p}$ a *multi-displacement* and $\mathbf{f}$ a *multi-force*. We can rewrite (4) as follows,

$$\mathbf{f} = -S\Delta\mathbf{p} \tag{5}$$

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \in \mathbb{R}^{2N\times 2N} \tag{6}$$

where S has components satisfying

$$\begin{cases} S_{ij} = -k_{ij}P_{ij} \quad (i \neq j) \\ S_{ii} = \sum_{m\in\mathcal{I}\setminus\{i\}} k_{im}P_{im} \end{cases} \quad i, j \in \mathcal{I} \tag{7}$$

Define S as the *stiffness matrix* of KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$. We may regard (5) as a generalized form of Hooke's Law due to its similarity to the expression $\mathbf{F} = -k\Delta\mathbf{x}$. The stiffness matrix S also has a structure resembling the Laplacian matrix of a graph.

If we assume that the system under perturbation remains stationary, by (5) it is clear that the perturbing multi-force $\tilde{\mathbf{f}}$ causing the displacement must satisfy the equilibrium condition

$$\tilde{\mathbf{f}} = -\mathbf{f} = S\Delta\mathbf{p}. \tag{8}$$

The increment of the elastic energy ΔJ, defined as the sum of elastic energy stored in each imaginary spring in the perturbed system, can be approximated as follows

$$\Delta J \doteq \int_0^1 \langle S(\alpha \cdot \Delta\mathbf{p}), \mathrm{d}\alpha \cdot \Delta\mathbf{p}\rangle = \frac{1}{2}\Delta\mathbf{p}^\top S \Delta\mathbf{p}. \tag{9}$$

IV. PROPERTIES OF STIFFNESS MATRIX

In this section, some attractive properties of the stiffness matrix defined by (6) and (7) are discussed.

A. Scale Invariance

For a given KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$, we may scale the configuration by a scalar $\gamma \in \mathbb{R}^+$ and form a new KP-formation, which can be described as $(\mathcal{I}, \gamma\hat{\mathbf{p}}, K)$. It is not difficult to show that the stiffness matrix S is invariant under scaling because for each $i, j \in \mathcal{I}, i \neq j$, P_{ij} is normalized, thus being constant with respect to γ.

B. Invariance under Global Rigid Motions

Given a 2×2 rotation matrix Q and a constant vector $\mathbf{c} \in \mathrm{R}^2$, a new configuration $\hat{\mathbf{p}}'$ is said to be transformed from $\hat{\mathbf{p}}$ under *global rigid motions* if $\hat{\mathbf{p}}'_i = Q\hat{\mathbf{p}}_i + \mathbf{c}$ for every $i \in \mathcal{I}$. From the isometric property of rotation and translation, we see that the two configurations $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$ are *congruent*, i.e., $\|\hat{\mathbf{p}}'_i - \hat{\mathbf{p}}'_j\| = \|\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j\|$ for every $i, j \in \mathcal{I}$.

Now we look at the new stiffness matrix S' under global rigid motions. The connectivity matrix K does not change, hence k_{ij} remains constant. The projection matrices P'_{ij} in S' are invariant with respect to translation $\mathbf{c}$, but the rotation operation will cause them to rotate accordingly, since

$$\begin{aligned} P'_{ij} = \mathbf{e}'_{ij}\mathbf{e}'^\top_{ij} &= \frac{1}{\|\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i\|^2} Q(\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i)(\hat{\mathbf{p}}_j - \hat{\mathbf{p}}_i)^\top Q^\top \\ &= QP_{ij}Q^\top. \end{aligned}$$

Therefore, we can represent the relation between S' and S by a similarity transformation,

$$S' = (Q \otimes I_N)S(Q \otimes I_N)^\top \tag{10}$$

where $\otimes$ denotes the Kronecker product of two matrices, I_N is the $N \times N$ identity matrix. Note that $Q \otimes I_N$ is a rotation matrix satisfying $(Q \otimes I_N)(Q \otimes I_N)^\top = I_{2N}$, which can be applied to the rotation of an N-member multi-point.

C. Non-negative Definiteness

The stiffness matrix S of a KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$ represents the linear relation between a set of perturbing forces and the resultant displacements on that formation. We may intuitively assert that S should have only non-negative eigenvalues, for perturbing forces $\tilde{\mathbf{f}}$ and resultant displacements $\Delta\mathbf{p}$ are unlikely to go in opposite directions. The proof of this intuition is given below.

Proof: Given any $\mathbf{v} = [\mathbf{v}_1^\top \;\; \mathbf{v}_2^\top \;\; \cdots \;\; \mathbf{v}_N^\top]^\top \in \mathbb{R}^{2N}$, where N is the number of agents, expand $\mathbf{v}^\top S\mathbf{v}$ using (6) and (7) and we have

$$\begin{aligned} \mathbf{v}^\top S\mathbf{v} &= \sum_{i,j\in\mathcal{I}} \mathbf{v}_i^\top S_{ij}\mathbf{v}_j \\ &= \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}\setminus\{j\}} k_{ij}\mathbf{v}_i^\top P_{ij}\mathbf{v}_i - \sum_{\substack{i,j\in\mathcal{I}\\ i\neq j}} k_{ij}\mathbf{v}_i^\top P_{ij}\mathbf{v}_j \end{aligned} \tag{11}$$

Recall that a projection matrix P_{ij} has the form $P_{ij} = \mathbf{e}_{ij}\mathbf{e}_{ij}^\top$. Therefore, (11) can be rewritten as follows,

$$\begin{aligned} \mathbf{v}^\top S\mathbf{v} &= \sum_{\substack{i,j\in\mathcal{I}\\ i\neq j}} k_{ij}\left(\mathbf{v}_i^\top\mathbf{e}_{ij} - \mathbf{v}_j^\top\mathbf{e}_{ij}\right)\left(\mathbf{e}_{ij}^\top\mathbf{v}_i - \mathbf{e}_{ij}^\top\mathbf{v}_j\right) \\ &= \sum_{\substack{i,j\in\mathcal{I}\\ i\neq j}} k_{ij}\left|(\mathbf{v}_i - \mathbf{v}_j)^\top\mathbf{e}_{ij}\right|^2 \end{aligned} \tag{12}$$

Hence, the stiffness matrix S is non-negative definite. ■

D. Relation to Rigidity Matrix

In rigidity theory, the concept of *rigidity matrix* is often useful to find out if a graph is rigid. [1] We hereby show the relation between stiffness matrix and rigidity matrix, and then adapt several conclusions from the properties of rigidity matrices due to their similarity.

The rigidity matrix is defined through an infinitesimal motion of the agents. Consider a KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$, if $k_{ij} > 0$ for some $i, j \in \mathcal{I}$, it implies that a distance constraint is active between point i and j, and therefore

$$\|\mathbf{p}_i - \mathbf{p}_j\|^2 \equiv d_{ij}^2$$

where p_i denotes the location of point i in a valid configuration. Now we take the derivative at the initial configuration $\hat{\mathbf{p}}$,

$$(\dot{\mathbf{p}}_i - \dot{\mathbf{p}}_j)^\top(\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j) = 0 \tag{13}$$

Let $\mathbf{w} = [\dot{\mathbf{p}}_1^\top \;\; \dot{\mathbf{p}}_2^\top \;\; \cdots \;\; \dot{\mathbf{p}}_N^\top]^\top \in \mathbb{R}^{2N}$. We can represent (13) in a matrix form,

$$R\mathbf{w} = \mathbf{0} \tag{14}$$

where $R \in \mathbb{R}^{M\times 2N}$ is called a *ridigity matrix*, and M is the number of active distance constraints. In rigidity theory, a configuration is called *infinitesimally rigid* if and only if $\mathrm{rank}(R) = 2N - 3$. [1], [2]

Now we consider the null space of S. Alternatively, the null space of S can be determined by $\mathrm{null}(S) = \{\mathbf{v} \in \mathbb{R}^{2N} : \mathbf{v}^\top S\mathbf{v} = 0\}$. From (12) it is clearly seen that $\mathbf{v} \in \mathrm{null}(s)$ if and only if the following holds,

$$(\mathbf{v}_i - \mathbf{v}_j)^\top\mathbf{e}_{ij} = 0, \forall\; i, j \in \mathcal{I} \text{ s.t. } i \neq j \text{ and } k_{ij} \neq 0 \tag{15}$$

Recall that $\mathbf{e}_{ij} = (\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i})/\|\hat{\mathbf{p}}_\mathbf{j} - \hat{\mathbf{p}}_\mathbf{i}\|$. We may find that the solutions to (15) coincides with those to (14), which reveals the fact that rigidity matrix R and stiffness matrix S share the same null space.

This property is of considerable significance because we can now claim that the KP-formation is rigid if the stiffness matrix S has rank $2N - 3$. This conclusion will be useful when we discuss the quantitative measurement of rigidity in the next section.

Also, from this relation one can claim that any global rigid motions (translation and rotation) fall into the null space of S, for it is known that they are within $\mathrm{null}(R)$. [9] We denote the set of two-dimensional global rigid motions $\mathbf{v}$ corresponding to a KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$ by $\mathrm{iso}_N(\hat{\mathbf{p}})$. It is not difficult to show that $\mathrm{iso}_N(\hat{\mathbf{p}})$ is a linear subspace of $\mathbb{R}^{2N}$ and the dimension of the subspace is 3.

V. QUANTITATIVE MEASURE OF RIGIDITY

In this section, a quantitative measure of KP-formation $(\mathcal{I}, \hat{\mathbf{p}}, K)$ is derived from the stiffness matrix S. We also show some interesting properties of this measure to demonstrate its value in practical applications.

A. Decomposition of Perturbation

For every formation, there exist global rigid motions that translate or rotate the formation as a whole without deformation. Configurations infinitesimally perturbed in the same directions of these global rigid motions are thus congruent to the original configuration. We may see that these kinds of perturbation will not change the inner energy related to the deformation but only the global kinetic energy, if the formation is regarded as a single rigid body. Therefore, it is necessary to exclude such perturbation when we analyze the rigidity of a formation.

In last section it is shown that the global rigid motions for a two-dimensional N-agent formation form a three-dimensional subspace $\mathrm{iso}_N(\hat{\mathbf{p}})$ of $\mathbb{R}^{2N}$. Therefore, for any $\mathbf{v} \in \mathbb{R}^{2N}$, there exist $\mathbf{u} \in \mathrm{iso}_N(\hat{\mathbf{p}})$ and $\mathbf{w} \in \mathrm{iso}_N(\hat{\mathbf{p}})^{\perp}$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. We call $\mathbf{u}$ *trivial perturbation* and $\mathbf{w}$ *equilibrium perturbation*.

However, for trivial perturbation $\Delta\mathbf{p} \in \mathrm{iso}_N(\hat{\mathbf{p}})$, we cannot assume that configuration $\mathbf{p}$ is congruent to $\mathbf{p} + \Delta\mathbf{p}$ because an element in $\mathrm{iso}_N(\hat{\mathbf{p}})$ represents only the direction of a rigid motion, not the result, i.e., the displacement of the motion. We may see that for $\Delta\mathbf{p}$ representing pure translations, $\mathbf{p}$ and $\mathbf{p}+\Delta\mathbf{p}$ are congruent; but if $\Delta\mathbf{p}$ involves rotation, the configuration is deformed. This, however, does not cause much trouble because by assumption the perturbing displacement is "very small", in which case we can let $\|\Delta\mathbf{p}\| \to 0$, which resolves the conflict in concepts.

Based on the above facts, we only consider perturbation $\Delta\mathbf{p} \in \mathrm{iso}_N(\hat{\mathbf{p}})^{\perp}$ in the following analysis. It can be verified that the rigidity indices are constants with respect to $\|\Delta\mathbf{p}\|$, hence it is valid to omit the analysis in limit case when $\|\Delta\mathbf{p}\| \to 0$.

B. Worst-Case Rigidity Index (W.R.I)

Recall that in (9) we gave the expression of the approximated energy increment ΔJ under a small perturbation $\Delta\mathbf{p}$. Now we fix the magnitude of the perturbation, i.e. $\|\Delta\mathbf{p}\| \equiv c$, and change the direction of perturbing displacements. Consequently, ΔJ varies with $\Delta\mathbf{p}$. The magnitude of ΔJ implies the sensitivity of the formation to the perturbation. With larger ΔJ, the agents are pushed back to the initial configuration harder, while $\Delta J = 0$ means the deformation does not raise any correction forces. The latter situation implies that the formation is not rigid under an infititesimal motion. Therefore, the worst case can be determined when ΔJ has the least magnitude. We define the *worst-case rigidity index* r_{w} to be twice the minimum ratio of ΔJ and $\|\Delta\mathbf{p}\|^2$,

$$r_{\mathrm{w}} \triangleq \min_{\Delta\mathbf{p} \notin \mathrm{iso}_N(\hat{\mathbf{p}})} \frac{2\Delta J}{\|\Delta\mathbf{p}\|^2} = \min_{\Delta\mathbf{p} \notin \mathrm{iso}_N(\hat{\mathbf{p}})} \frac{\Delta\mathbf{p}^\top S \Delta\mathbf{p}}{\Delta\mathbf{p}^\top \Delta\mathbf{p}} \tag{16}$$

The coefficient 2 here is inspired by $k = 2E/(\Delta x)^2$ for a single-spring system.

Since S is real symetric non-negative definite matrix, S has $2N$ non-negative eigenvalues $\lambda_n, n = 1, 2, \cdots, 2N$. We sort them such that $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2N}$. As it is already shown that $\mathrm{iso}_N(\hat{\mathbf{p}}) \subset \mathrm{null}(S)$ and $\dim(\mathrm{iso}_N(\hat{\mathbf{p}})) = 3$, the minimization over the space $\mathbb{R}^{2N} \backslash \mathrm{iso}_N(\hat{\mathbf{p}})$ gives us the fourth smallest eigenvalue, hence

$$r_{\mathrm{w}} = \lambda_4 \tag{17}$$

C. Properties of Worst-Case Rigidity Index

We hereby show some properties of worst-case rigidity index. These properties conform to our intuitive assumptions of a rigidity measure.

Property 1. A non-rigid graph has zero as the value of its worst-case rigidity index. This is a corollary from (17), since a non-rigid graph has a rigidity matrix R such that $\mathrm{rank}(R) \le 2N - 4$, therefore $\dim(\mathrm{null}(R)) \ge 4$. Recalling the fact that $\mathrm{null}(R) = \mathrm{null}(S)$ where S is the stiffness matrix, we conclude that $\lambda_4 = 0$.

Property 2. The worst-case rigidity index is an invariant under rigid body motions and scaling of the formation. This is a direct result from the properties of the stiffness matrix shown in Section IV. Recall that stiffness matrix S is invariant under scaling and translation, and rigid body rotations only result a similarity transformation which does not change the eigenvalues of a matrix. Therefore, r_{w} is invariant under these operations.

Property 3. r_{w} is a monotone increasing function of $k_{ij}, \forall i, j \in \mathcal{I}$. Its interpretation is that, increasing the connectivity between any pair of nodes will always contribute positively to the global rigidity. This property well fits our intuitive assumption of a rigidity measure.

Proof: Suppose we increment the connectivity matrix K to $K' = [k'_{ij}]$, in the sense that $k'_{ij} = k_{ij} + \Delta k_{ij}, \Delta k_{ij} = \Delta k_{ji} > 0, \forall\ i, j \in \mathcal{I}, i \ne j$. Let S be the stiffness matrix of $(\mathcal{I}, \hat{\mathbf{p}}, K)$ and S' be the stiffness matrix of $(\mathcal{I}, \hat{\mathbf{p}}, K')$. The worst-case rigidity indices are r_{w} and r'_{w} accordingly. By linearity, we may find that $S' - S$ actually satisfies the definition of the stiffness matrix of $(\mathcal{I}, \hat{\mathbf{p}}, K' - K)$. Therefore, $S' - S \ge 0$ by property of stiffness matrix. This is equivalent to $S' \ge S$. Note that S and S' share the same rigid motion space $\mathrm{iso}_N(\hat{\mathbf{p}})$, by definition of worst-case rigidity index in (16), we can readily reach the conclusion that $r'_{\mathrm{w}} \ge r_{\mathrm{w}}$. ■

The above properties ensure the validity of the worst-case rigidity index as a quantitative measure of formation rigidity.

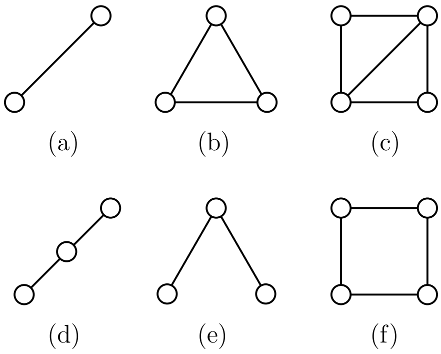

Fig. 2. Examples of simple formations

From the definition we may see a larger r_{w} indicates greater inflexibility of a formation. This then can be used in the applications where comparisons between different formations are demanded.

VI. EXAMPLES

In Section V, two measures of formation rigidity are proposed: worst-case rigidity index (W.R.I) and mean rigidity index (M.R.I). In this section, some examples are given to illustrate quantitative analysis of a formation using these rigidity indices.

A. Rigid and Non-Rigid Formations

We first look at some simple formations illustrated in Fig. 2, where a circle represents an agent, and an egde between a pair of agents represents a distance constraint with connectivity coefficient fixed to 1. Table I gives the rigidity indices of these formations.

TABLE I

RIGIDITY INDICES OF FORMATIONS IN FIG. 2

Formation	No. of agents	Is rigid?	W.R.I
a	2	Yes	2
b	3	Yes	1.5
c	4	Yes	0.586
d	3	No	0
e	3	No	0
f	4	No	0

In compliance to the results in Section V-C, the values of worst-case rigidity index for non-rigid formations are zero.

B. Rigid Formations with Same Number of Agents

Next, we try to utilize the worst-case rigidity index to compare among several formations that have the same number of agents. We still assume that every distance constraint has an equal connectivity weight, in which case we may let $k_{ij} = 1$ distance constraint is applied between agent i and j without loss of generality. The sample formations are illustrated in Fig. 3. Each formation contains five agents. Formation g_1,g_2,g_3 share the same configuration but differ in the number of distance constraints. Table II lists the results of the quantitative rigidity tests.

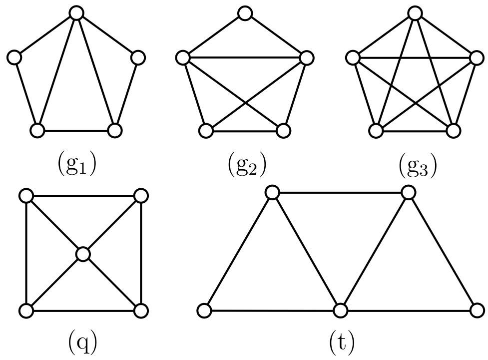

Fig. 3. Examples of rigid formations with 5 agents

TABLE II

RIGIDITY INDICES OF FORMATIONS IN FIG. 3

Formation	No. of agents	Is rigid?	W.R.I
g_1	5	Yes	0.4912
g_2	5	Yes	0.4939
g_3	5	Yes	2.5
q	5	Yes	1
t	5	Yes	0.5829

Comparing Formation (g_1),(g_2),(g_3), we may clearly see that with the incrementing number of distance constraints, the worst-case rigidity index rises accordingly, which satisfy the argument in Section V-C. Notice that from (g_1) to (g_2), the increase of the value is small, while (g_3) has a much larger value of the rigidity index. This implies that symmetry of a formation may have great positive impact on formation rigidity, for it usually has the least vulnerability in all directions of perturbation.

We can also compare Formation (g_2) with (q) since their number of distance constraints are also equal. Result shows that Formation (q), which has a symmetric structure, owns a much higher value of worst-case rigidity index than (g_2). From comparison between Formation (g_1) and (t), we may assert that Formation (t) is likely to outperform (g_1) in most applications regarding multi-agent formation control. An brief explanation can be the fact that equilateral triangles are more stable than non-equilateral ones.

VII. CONCLUSION

In this paper, we have come up with two new concepts of a formation, the stiffness matrix and the worst-case rigidity index, where the latter value is a scalar derived from the stiffness matrix. Further calculation reveals that the worst-case rigidity index coincides with the fourth smallest eigenvalue of stiffness matrix. We also showed that this rigidity index has properties that meet the demands in real applications, hence being a practical quatitative measure of formation rigidity.

By using the worst-case rigidity index, we have noticed an interesting result in Section VI that symmetric formations are usually far more rigid than asymmetric formations, in the sense that the former usually has a worst-case rigidity index much higher than the latter, given the same number of agents and distance constraints in both formations. This information may become helpful when one designs a formation for multi-agent control system.

There are several perspective topics related to the quantitative measurement of formation rigidity. First, we would like to know if the conclusions can be generalized to three or higher dimensional formations, and what additional properties of either stiffness matrix or worst-case rigidity index can be found in such higher dimensional spaces. Second, we want to search for a measurement that can compare two non-rigid structure, since the worst-case rigidity index will be zero for every formation that is not infinitesimally rigid. Finally, the worst-case rigidity index only characterize the rigidity of a static formation. We want to know if moving formations will have extra dynamic information which can be extracted to determine the quantitative rigidity.

REFERENCES

[1] W. Whiteley, "Some matroids from discrete applied geometry," in *Matroid Theory: Ams-IMS-Siam Joint Summer Research Conference on Matroid Theory, July 2-6, 1995, University of Washington, Seattle*, vol. 197. American Mathematical Society, 1996, p. 171.

[2] B. Jackson and T. Jordan, "Connected rigidity matroids and unique realizations of graphs," *Journal of Combinatorial Theory, Series B*, vol. 94, no. 1, pp. 1–29, 2005.

[3] G. Laman, "On graphs and rigidity of plane skeletal structures," *Journal of Engineering mathematics*, vol. 4, no. 4, pp. 331–340, 1970.

[4] B. Anderson, C. Yu, B. Fidan, and J. Hendrickx, "Rigid graph control architectures for autonomous formations," *Control Systems Magazine, IEEE*, vol. 28, no. 6, pp. 48–63, Dec. 2008.

[5] R. Olfati-Saber and R. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems," *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 3, pp. 2965–2971, Dec. 2002.

[6] B. Fidan, C. Yu, and B. Anderson, "Acquiring and maintaining persistence of autonomous multi-vehicle formations," *IET Control Theory and Applications*, vol. 1, no. 2, pp. 452–460, 2007.

[7] J. Fang and A. Morse, "Network localization using graph decomposition and rigidity," *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pp. 1091–1096, Dec. 2008.

[8] J. Aspnes, T. Eren, D. Goldenberg, A. Morse, W. Whiteley, Y. Yang, B. Anderson, and P. Belhumeur, "A theory of network localization," *IEEE Transactions on Mobile Computing*, vol. 5, no. 12, pp. 1663–1678, 2006.

[9] B. Hendrickson, "Conditions for unique graph realizations," *SIAM J. Comput.*, vol. 21, no. 1, pp. 65–84, 1992.