A Generating Function Approach to the Stability of Discrete-Time Switched Linear Systems*

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ABSTRACT

Exponential stability of switched linear systems under both arbitrary and proper switching is studied through two suitably defined families of functions called the strong and the weak generating functions. Various properties of the generating functions are established. It is found that the radii of convergence of the generating functions characterize the exponential growth rate of the trajectories of the switched linear systems. In particular, necessary and sufficient conditions for the exponential stability of the systems are derived based on these radii of convergence. Numerical algorithms for computing estimates of the generating functions are proposed and examples are presented for illustration purpose.

Categories and Subject Descriptors

G.1.0 [**Numerical Analysis**]: General—*stability (and instability), numerical algorithms*; I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*control theory,dynamic programming*

General Terms

Theory, Algorithms

Keywords

Switched linear systems, exponential stability, generating functions.

1. INTRODUCTION

Switched linear systems are a natural extension of linear systems and an important family of hybrid systems. They have been finding increasing applications in a diverse range of engineering systems [11]. A fundamental problem in the study of switched linear systems is to determine their stability, or more generally, to characterize the exponential rate at which their trajectories grow or decay, under various switching rules. See [13, 15] for some recent reviews of the vast amount of existing work on this subject. Specifically, these work can be classified into two categories: stability under arbitrary switching where the switching rules are unconstrained; and stability under restricted switching rules such as switching rate constraints [6] and state-dependent switchings [9]. A predominant approach to the study of stability in both cases is through the construction of common or multiple Lyapunov functions [3, 8, 9]. Other approaches include Lie algebraic conditions [12] and the LMI methods.

The goal of this paper is to determine not only the exponential stability of switched linear systems, but also the exponential growth rate of their trajectories. In particular, we try to characterize the maximum exponential growth rate of the trajectories under arbitrary switching and the minimum exponential growth rate under proper switching. In the latter case, the switching is fully controllable; thus the problem can be deemed as a switching stabilization problem. Previous contributions in the literature on these two rates, especially the first one, include for example the work on joint spectral radius [2, 16, 17] and (maximum) Lyapunov exponent [1], to name a few. These work studied directly the maximum growth rate of the norm of increasingly longer products of subsystem matrices.

In comparison, the method proposed in this paper characterizes the exponential growth rates indirectly through two families of functions, the strong and the weak generating functions, that are power series with coefficients determined by the system trajectories. The advantages of such a method are: (i) quantities derived from the generating functions, such as their radii of convergence and quadratic bounds, fully characterize the exponential growth of the system trajectories, including but not limited to the exponential growth rates; (ii) these functions are automatically Lyapunov functions if the systems are exponentially stable; (iii) the generating functions possess many nice properties that make their efficient computation possible; (iv) and finally, such a method admits natural extensions to more general classes of systems, such as conewise linear inclusions [14] and switched linear systems with control input.

This paper is organized as follows. In Section 2, the stability notions of switched linear systems are briefly reviewed. In Section 3, we define the strong generating functions; study their various properties; and use them to characterize the ex-

*An extended and improved version of the results in this paper has been submitted as [7].

HSCC'10, April 12–15, 2010, Stockholm, Sweden.

ponential stability of the systems under arbitrary switching. A numerical algorithm will also be presented to compute the strong generating functions by taking advantage of their convexity. The development of Section 4 mirrors that of Section 3, where weak generating functions are studied for the purpose of characterizing the exponential stability of the systems under proper switching. Finally, some concluding remarks are given in Section 5.

2. STABILITY OF SWITCHED LINEAR SYSTEMS

A discrete-time (autonomous) switched linear system (SLS) is defined as follows: its state $x(t) \in \mathbb{R}^n$ evolves by switching among a set of linear dynamics indexed by the finite index set $\mathcal{M} := \{1, \dots, m\}$:

$$x(t+1) = A_{\sigma(t)} x(t), \quad t = 0, 1, \dots. \tag{1}$$

Here, $\sigma(t) \in \mathcal{M}$ for $t = 0, 1 \dots$, or simply σ, is called the switching sequence; and $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{M}$, are the subsystem state dynamics matrices (assume $A_i \neq 0$). Starting from the initial state $x(0) = z$, the trajectory of the SLS under the switching sequence σ will be denoted by $x(t; z, \sigma)$.

Definition 1. *The switched linear system is called*

- exponentially stable under arbitrary switching *(with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any initial state z and under any switching sequence σ, the trajectory $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, for all $t = 0, 1, \dots$.*
- exponentially stable under proper switching *(with the parameters κ and r) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any initial state z, there exists a switching sequence σ for which the trajectory $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, for all $t = 0, 1, \dots$.*

Similar to linear systems, we can define the notions of stability (in the sense of Lyapunov) and asymptotic stability for SLS under both arbitrary and proper switching. Due to the homogeneity of SLS, the local and global versions of these stability notions are equivalent. Moreover, it is easily shown that the asymptotic stability and exponential stability of SLS under arbitrary switching are equivalent [14].

3. STRONG GENERATING FUNCTIONS

Central to the stability analysis of SLS is the task of determining the exponential rate at which $\|x(t; z, \sigma)\|$ grows as $t \to \infty$ for trajectories $x(t; z, \sigma)$ of the SLS. The following lemma, adopted from [10, Corollary 1.1.10], hints at an indirect way of characterizing this growth rate.

Lemma 1. *For a given sequence of scalars $\{a_t\}_{t=0,1,\dots}$, suppose the power series $\sum_{t=0}^{\infty} a_t \lambda^t$ has the radius of convergence R. Then for any $r > \frac{1}{R}$, there exists a constant C_r such that $|a_t| \leq C_r r^t$ for all $t = 0, 1, \dots$.*

As a result, for any trajectory $x(t; z, \sigma)$ of the SLS, an (asymptotically) tight bound on the exponential growth rate of $\|x(t; z, \sigma)\|^2$ as $t \to \infty$ is given by the reciprocal of the radius of convergence of the power series $\sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^2$. This motivates the following definition in studying the exponential stability of SLS under arbitrary switching.

For each $z \in \mathbb{R}^n$, define the *strong generating function* $G(\cdot, z) : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ of the SLS as

$$G(\lambda, z) := \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|x(t, z, \sigma)\|^2, \quad \forall \lambda \geq 0, \tag{2}$$

where the supremum is taken over all switching sequences σ of the SLS. Obviously, $G(\lambda, z)$ is monotonically increasing in λ, with $G(0, z) = \|z\|^2$. As λ increases, however, it is possible that $G_\lambda(z) = +\infty$. Define the threshold

$$\lambda^*(z) := \sup\{\lambda \mid G(\lambda, z) < +\infty\}$$

as the *radius of strong convergence* of the SLS at z. Intuitively speaking, $G(\cdot, z)$ corresponds to the power series of the "most divergent" trajectories starting from z generated by all switching sequences; thus the radius of convergence $\lambda^*(z)$ is expected to contain information about the fastest exponential growth rate for solutions starting from z.

It is often convenient to study $G(\lambda, z)$ as a function of z for a fixed λ. Thus, for each $\lambda \geq 0$, define the function $G_\lambda : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ as

$$G_\lambda(z) := G(\lambda, z), \quad \forall z \in \mathbb{R}^n. \tag{3}$$

From its definition, $G_\lambda(z)$ is nonnegative, homogeneous of degree two in z, with $G_0(z) = \|z\|^2$. We will also refer to $G_\lambda(z)$ as the strong generating function of the SLS.

3.1 Properties of General $G_\lambda(z)$

Some properties of the function $G_\lambda(z)$ are listed below.

Proposition 1. *$G_\lambda(z)$ has the following properties.*

1. *(Bellman Equation): For all $\lambda \geq 0$ and all $z \in \mathbb{R}^n$, we have $G_\lambda(z) = \|z\|^2 + \lambda \cdot \max_{i \in \mathcal{M}} G_\lambda(A_i z)$.*
2. *(Sub-Additivity): Let $\lambda \geq 0$ be arbitrary. Then*
$$\sqrt{G_\lambda(z_1 + z_2)} \leq \sqrt{G_\lambda(z_1)} + \sqrt{G_\lambda(z_2)}$$
for all $z_1, z_2 \in \mathbb{R}^n$. Hence, for $\alpha_1, \alpha_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^n$, we have $G_\lambda(\alpha_1 z_1 + \alpha_2 z_2) \leq 2\alpha_1^2 G_\lambda(z_1) + 2\alpha_2^2 G_\lambda(z_2)$.
3. *(Convexity): For each $\lambda \geq 0$, both $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ are convex functions of $z \in \mathbb{R}^n$.*
4. *(Invariant Subspace): For each $\lambda \geq 0$, the set $\mathcal{G}_\lambda := \{z \in \mathbb{R}^n \mid G_\lambda(z) < +\infty\}$ is a subspace of $\mathbb{R}^n$ invariant under $\{A_i\}_{i \in \mathcal{M}}$, namely, $A_i \mathcal{G}_\lambda \subset \mathcal{G}_\lambda$ for all $i \in \mathcal{M}$.*
5. *For $0 \leq \lambda < (\max_{i \in \mathcal{M}} \|A_i\|^2)^{-1}$, $G_\lambda(z) < \infty$ for all z.*
6. *If $\lambda \geq 0$ is such that $G_\lambda(z) < +\infty$, $\forall z \in \mathbb{R}^n$, then there exists a constant $c \in [1, \infty)$ such that $\|z\|^2 \leq G_\lambda(z) \leq c\|z\|^2$, $\forall z \in \mathbb{R}^n$.*

Proof. 1. Property 1 is a direct consequence of the dynamic programming principle if we view $G_\lambda(z)$ as the value function of an infinite horizon optimal control problem.

2. Since $x(t;z,\sigma)$ is linear in z, $\forall \lambda \ge 0$, $\forall z_1, z_2 \in \mathbb{R}^n$,

$$\begin{aligned}
G_\lambda(z_1+z_2) &= \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;z_1,\sigma)+x(t;z_2,\sigma)\|^2 \\
&\le \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \big[\|x(t;z_1,\sigma)\|^2 + 2\|x(t;z_1,\sigma)\|\cdot\|x(t;z_2,\sigma)\| \\
&\qquad + \|x(t;z_2,\sigma)\|^2\big] \\
&\le G_\lambda(z_1) + 2\sqrt{G_\lambda(z_1)}\sqrt{G_\lambda(z_2)} + G_\lambda(z_2) \\
&= \left[\sqrt{G_\lambda(z_1)} + \sqrt{G_\lambda(z_2)}\right]^2,
\end{aligned}$$

which is the first desired conclusion. Using the Cauchy-Schwartz inequality, we can derive the second desired conclusion from the first one.

3. For each fixed $\lambda \ge 0$, the convexity of $\sqrt{G_\lambda(z)}$ follows from its sub-additivity: for arbitrary $z_1, z_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \ge 0$ with $\alpha_1 + \alpha_2 = 1$,

$$\begin{aligned}
\sqrt{G_\lambda(\alpha_1 z_1 + \alpha_2 z_2)} &\le \sqrt{G_\lambda(\alpha_1 z_1)} + \sqrt{G_\lambda(\alpha_2 z_2)} \\
&= \alpha_1\sqrt{G_\lambda(z_1)} + \alpha_2\sqrt{G_\lambda(z_2)}.
\end{aligned}$$

As a result, $G_\lambda(z) = (\sqrt{G_\lambda(z)})^2$ is also convex.

4. The conclusions follow directly from sub-additivity and the Bellman equation of $G_\lambda(z)$.

5. For any trajectory $x(t;z,\sigma)$ of the SLS, simply observe that $\|x(t;z,\sigma)\|^2 \le (\max_{i\in\mathcal{M}} \|A_i\|^2)^t \|z\|^2$ for all t.

6. Assume λ is such that $G_\lambda(z)$ is finite for all z. By its definition, $G_\lambda(z) \ge \|z\|^2$. To show that $G_\lambda(z) \le c\|z\|^2$, by homogeneity it suffices to show that $G_\lambda(z) \le c$ for all z on the unit sphere $\mathbb{S}^{n-1}$. Each $z \in \mathbb{S}^{n-1}$ can be written as $z = \sum_{i=1}^n \alpha_i\, \mathbf{e}_i$, where $\{\mathbf{e}_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$, and $\{\alpha_i\}_{i=1}^n$ are the coordinates of z in this basis satisfying $\sum_{i=1}^n \alpha_i^2 = 1$. In light of sub-additivity, $G_\lambda(z) \le n\sum_{i=1}^n \alpha_i^2 G_\lambda(\mathbf{e}_i) \le c$, where $c = n \cdot \max_{1\le i\le n} G_\lambda(\mathbf{e}_i) < \infty$ by our assumption on λ. This completes the proof. □

From Proposition 1, $\mathcal{G}_\lambda$ is a subspace of $\mathbb{R}^n$ that decreases monotonically from $\mathcal{G}_0 = \mathbb{R}^n$ at $\lambda = 0$ to $\mathcal{G}_\infty := \cap_{\lambda\ge 0}\mathcal{G}_\lambda$ as $\lambda \to \infty$. The set of all distinct $\mathcal{G}_\lambda$ for $\lambda \ge 0$ forms a cascade of subspaces of $\mathbb{R}^n$: $\mathbb{R}^n = \mathcal{G}_{\lambda_1} \supsetneq \cdots \supsetneq \mathcal{G}_{\lambda_d}$ for some $0 = \lambda_1 < \cdots < \lambda_d$, where $d \le n$ is an integer. Since each of such $\mathcal{G}_{\lambda_j}$ is invariant under $\{A_i\}_{i\in\mathcal{M}}$, any trajectory of the SLS starting from an initial state in $\mathcal{G}_{\lambda_j}$ will remain inside $\mathcal{G}_{\lambda_j}$ at all subsequent times. Thus, a sub-SLS can be defined as the restriction of the original SLS on the subspace $\mathcal{G}_{\lambda_j}$. Intuitively, the restricted sub-SLS on $\mathcal{G}_{\lambda_d}$ is the "most exponentially stable" sub-SLS as its trajectories have the largest radius of convergence, hence the slowest exponential growth rate. As j decreases, the restricted sub-SLS on $\mathcal{G}_{\lambda_j}$ will become "less exponentially stable" as faster growing trajectories are included.

The above geometric statements can also be stated equivalently as follows: after a suitable change of coordinates, all the matrices A_i, $i \in \mathcal{M}$, can be simultaneously transformed into the same row block upper echelon form, with their last row blocks corresponding to the restricted sub-SLS on $\mathcal{G}_{\lambda_d}$; their last two row blocks corresponding to the restricted sub-SLS on $\mathcal{G}_{\lambda_{d-1}}$, ... etc. Finally, if the SLS is *irreducible*, namely, has no nontrivial invariant subspaces other than $\mathbb{R}^n$ and $\{0\}$ (which occurs with probability one for randomly generated SLS), the above discussions imply that the function $G_\lambda(z)$ is either finite everywhere or infinite everywhere for any given $\lambda \ge 0$.

Example 1. *Consider the following SLS on* $\mathbb{R}^2$*:*

$$A_1 = \begin{bmatrix} \frac{7}{6} & -\frac{5}{6} \\ -\frac{5}{6} & \frac{7}{6} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{5}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix}. \tag{4}$$

Starting from any initial $z = (z_1, z_2)^T$*, let* $x(t;z,\sigma_1)$ *and* $x(t;z,\sigma_2)$ *be the state trajectories under the switching sequences* $\sigma_1 = (1,1,\ldots)$ *and* $\sigma_2 = (2,2,\ldots)$*, respectively. Then it can be proved (though by no mean trivially) that*

$$\begin{aligned}
G_\lambda(z) &= \max\left\{\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma_1)\|^2, \sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma_2)\|^2\right\} \\
&= \begin{cases}
\max\left\{\frac{9(z_1+z_2)^2}{2(9-\lambda)} + \frac{(z_1-z_2)^2}{2(1-4\lambda)},\right. \\
\qquad \left.\frac{(z_1+z_2)^2}{2(1-9\lambda)} + \frac{9(z_1-z_2)^2}{2(9-\lambda)}\right\}, & \text{if } 0 \le \lambda < \frac{1}{9} \\
\frac{(z_1-z_2)^2}{2(1-4\lambda)} \cdot 1_{z_1+z_2=0} + \infty \cdot 1_{z_1+z_2\ne 0}, & \text{if } \frac{1}{9} \le \lambda < \frac{1}{4} \\
\infty, & \text{if } \lambda \ge \frac{1}{4}.
\end{cases}
\end{aligned}$$

Here, $1_{z_1+z_2=0}$ *is the indicator function for the set* $\{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 + z_2 = 0\}$*, etc. Thus,* $\mathcal{G}_\lambda$ *is* $\mathbb{R}^2$ *for* $0 \le \lambda < \frac{1}{9}$*; span*$\{(1,-1)^T\}$ *for* $\frac{1}{9} \le \lambda < \frac{1}{4}$*; and* $\{0\}$ *for* $\lambda \ge \frac{1}{4}$*. Each of these, e.g., span*$\{(1,-1)^T\}$*, is an invariant subspace of* $\mathbb{R}^2$ *for* $\{A_1, A_2\}$*. Indeed, one can verify that the two system matrices* A_1 *and* A_2 *can be simultaneously diagonalized by the transformation matrix* $Q = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$ *as* $Q^T A_1 Q = \mathrm{diag}(\frac{1}{3}, 2)$ *and* $Q^T A_2 Q = \mathrm{diag}(3, \frac{1}{3})$*, respectively.*

3.2 Radius of Strong Convergence

The quantity defined below will be important in characterizing the stability of the SLS under arbitrary switching.

Definition 2. *The radius of strong convergence of the SLS (1) is the quantity* $\lambda^* \in (0,\infty]$ *defined by*

$$\begin{aligned}
\lambda^* := \sup\big\{\lambda \mid & \text{ there exists a finite constant } c \\
& \text{ such that } G_\lambda(z) < c\|z\|^2,\ \forall z \in \mathbb{R}^n\big\}.
\end{aligned}$$

By Property 6 of Proposition 1, λ^* can also be defined by $\lambda^* = \sup\{\lambda \mid G_\lambda(z) < \infty,\ \forall z \in \mathbb{R}^n\}$. By Property 5, $\lambda^* \ge (\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1} > 0$. It is possible that $\lambda^* = +\infty$. This is the case, for example, if all solutions $x(t;z,\sigma)$ of the SLS converge to the origin within a finite time independent of the starting state z. For the SLS in Example 1, its radius of strong convergence is $\frac{1}{9}$.

The following theorem shows that the knowledge of the radius of strong convergence is sufficient for determining the stability of the SLS under arbitrary switching.

Theorem 1. *The following statements are equivalent:*

1. *The SLS is exponentially stable under arbitrary switching.*
2. *Its radius of strong convergence* $\lambda^* > 1$*.*
3. *The generating function* $G_1(z)$ *is finite for all* $z \in \mathbb{R}^n$*.*

Proof. $1 \Rightarrow 2$: Suppose there exist constants $\kappa \ge 1$ and $r \in [0,1)$ such that $\|x(t;z,\sigma)\| \le \kappa r^t \|z\|$, $t = 0, 1, \ldots$, for all trajectory $x(t;z,\sigma)$ of the SLS. Then for any $\lambda < r^{-2}$,

$$G_\lambda(z) = \sup_\sigma \sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma)\|^2 \le \frac{\kappa^2}{1-\lambda r^2}\|z\|^2,$$

is finite for all z. It follows that $\lambda^* \geq r^{-2} > 1$.

$2 \Rightarrow 3$: This follows directly from the definition of λ^*.

$3 \Rightarrow 1$: Suppose $G_1(z)$ is finite for all z. By Property 6 of Proposition 1, $G_1(z) \leq c\|z\|^2$ for some constant $c < +\infty$. Thus, for any trajectory $x(t; z, \sigma)$ of the SLS, $\sum_{t=0}^{\infty} \|x(t; z, \sigma)\|^2 \leq c\|z\|^2$. This implies that $\|x(t; z, \sigma)\| \leq \sqrt{c}\|z\|$ for all $t = 0, 1, \ldots$, namely, the SLS is stable under arbitrary switchings; and that $x(t; z, \sigma) \to 0$ as $t \to \infty$. Consequently, the SLS is asymptotically, hence exponentially, stable under arbitrary switching. □

This implies the following stronger conclusions.

Corollary 1. *Given a SLS with a radius of strong convergence λ^*, for any $r > (\lambda^*)^{-1/2}$, there exists a constant κ_r such that $\|x(t; z, \sigma)\| \leq \kappa_r r^t \|z\|$, $t = 0, 1, \ldots$, for all trajectories $x(t; z, \sigma)$ of the SLS.*

Proof. Let $r > (\lambda^*)^{-1/2}$ be arbitrary. The scaled SLS with subsystem dynamics matrices $\{A_i/r\}_{i \in \mathcal{M}}$ is easily seen to have its strong generating function to be $G(\lambda/r^2, z)$; hence its has a radius of strong convergence $r^2\lambda^* > 1$. By Theorem 1, the scaled SLS is exponentially stable under arbitrary switching; in particular, all its trajectories $\tilde{x}(t; z, \sigma)$ satisfy $\|\tilde{x}(t; z, \sigma)\| \leq \kappa_r \|z\|$, $t = 0, 1, \ldots$, for some $\kappa_r > 0$. Note that trajectories $\tilde{x}(t; z, \sigma)$ of the scaled SLS are exactly $r^{-t}x(t; z, \sigma)$ where $x(t; z, \sigma)$ are the trajectories of the original SLS. Thus, $\|x(t; z, \sigma)\| = r^t \|\tilde{x}(t; z, \sigma)\| \leq \kappa_r r^t \|z\|$, $t = 0, 1, \ldots$, for all trajectories $x(t; z, \sigma)$ of the original SLS. □

Thus, the fastest exponential growth rate r for all trajectories of the SLS can be chosen to be arbitrarily close to $(\lambda^*)^{-1/2}$. Later on in Theorem 2 in Section 3.5, the constant κ_r will be derived from the strong generating functions.

3.3 Smoothness of Finite $G_\lambda(z)$

When $\lambda \in [0, \lambda^*)$, the function $G_\lambda(z)$ is finite everywhere. From this point on, we shall focus on such finite $G_\lambda(z)$. First, some smoothness properties are established.

We first introduce a few notions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *directionally differentiable* at $z_0 \in \mathbb{R}^n$ if its (one-sided) directional derivative at z_0 along $v \in \mathbb{R}^n$ defined as $f'(z_0; v) := \lim_{\tau \downarrow 0} \frac{f(z_0 + \tau v) - f(z_0)}{\tau}$ exists for any direction v. If f is both directionally differentiable at z_0 and locally Lipschitz continuous in a neighborhood of z_0, it is called *B(ougligand)-differentiable* at z_0. Finally, the function f is *semismooth* at z_0 if it is B-differentiable in a neighborhood of z_0 and the following limit holds:

$$\lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{|f'(z; z - z_0) - f'(z_0; z - z_0)|}{\|z - z_0\|} = 0.$$

Proposition 2. *Let $\lambda \in [0, \lambda^*)$ be arbitrary.*

1. *Both $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ are convex, locally Lipschitz continuous functions on $\mathbb{R}^n$. Moreover, $\sqrt{G_\lambda(z)}$ is globally Lipschitz continuous.*
2. *Both $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ are semismooth on $\mathbb{R}^n$.*

Proof. 1. The convexity has been proved in Proposition 1. To show the global Lipschitz continuity of $\sqrt{G_\lambda(z)}$, we invoke its sub-additivity to obtain, for any $z, \Delta z \in \mathbb{R}^n$,

$$-\sqrt{G_\lambda(-\Delta z)} \leq \sqrt{G_\lambda(z + \Delta z)} - \sqrt{G_\lambda(z)} \leq \sqrt{G_\lambda(\Delta z)}.$$

Hence, $\left|\sqrt{G_\lambda(z + \Delta z)} - \sqrt{G_\lambda(z)}\right| \leq \sqrt{G_\lambda(\pm \Delta z)} \leq \sqrt{c}\|\Delta z\|$ for some constant c as $\lambda < \lambda^*$. This shows that $\sqrt{G_\lambda(z)}$ is globally Lipschitz continuous on $\mathbb{R}^n$ with the Lipschitz constant $\sqrt{c}$. As a result, $\sqrt{G_\lambda(z)}$, hence $G_\lambda(z)$, is (locally Lipschitz) continuous on $\mathbb{R}^n$.

2. The semismoothness of $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ follows directly from their convexity and local Lipschitz continuity. Indeed, any convex and locally Lipschitz continuous function f on $\mathbb{R}^n$ must be directionally differentiable, hence B-differentiable, on $\mathbb{R}^n$. To see this, note that, for any fixed $z_0, v \in \mathbb{R}^n$, the function $g(\tau) := \frac{f(z_0 + \tau v) - f(z_0)}{\tau}$ is non-decreasing in $\tau > 0$ by the convexity of f; and bounded from below as $\tau \downarrow 0$ by the local Lipschitz continuity of f. Thus, $f'(z_0, v) = \lim_{\tau \downarrow 0} g(\tau)$ exists. We claim further that such an f must also be semismooth on $\mathbb{R}^n$. A proof of this claim using the equivalent formulation of semismoothness in term of Clarke's generalized gradient [4] can be found in [5, Proposition 7.4.5]. This completes the proof of the semismoothness of $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ on $\mathbb{R}^n$. □

It is worth pointing out that if $\lambda \geq \lambda^*$ is outside the range of $[0, \lambda^*)$, then $G_\lambda(z)$ can not be continuous on $\mathbb{R}^n$. Indeed, it is not even continuous at $z = 0$.

3.4 Quadratic Bound of Finite $G_\lambda(z)$

For each $\lambda \in [0, \lambda^*)$, $G_\lambda(z)$ is finite everywhere, hence quadratically bounded by Proposition 1. Define the constant

$$g_\lambda := \sup_{\|z\|=1} G_\lambda(z), \quad \lambda \in [0, \lambda^*). \tag{5}$$

Then g_λ is finite and strictly increasing on $[0, \lambda^*)$. By homogeneity, g_λ can be equivalently defined as the smallest constant c such that $G_\lambda(z) \leq c\|z\|^2$ for all $z \in \mathbb{R}^n$. By continuity of $G_\lambda(z)$ proved in Proposition 2, for each $\lambda \in [0, \lambda^*)$, $G_\lambda(z) = g_\lambda \|z\|^2$ for some $z \in \mathbb{R}^n$.

The following estimate of g_λ can be easily obtained.

Lemma 2. $\frac{1}{g_\lambda} \geq 1 - \lambda \cdot \max_{i \in \mathcal{M}} \|A_i\|^2$, *for $\lambda \in [0, \lambda^*)$.*

Proof. Let $\lambda \in [0, \lambda^*)$. For any trajectory $x(t; z, \sigma)$ of the SLS, we have $\|x(t; z, \sigma)\|^2 \leq \max_{i \in \mathcal{M}} \|A_i\|^2 \cdot \|x(t - 1; z, \sigma)\|^2 \leq \cdots \leq (\max_{i \in \mathcal{M}} \|A_i\|^2)^t \|z\|^2$. Thus,

$$\sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^2 \leq \frac{1}{1 - \lambda \cdot \max_{i \in \mathcal{M}} \|A_i\|^2} \|z\|^2,$$

for $0 \leq \lambda < (\max_{i \in \mathcal{M}} \|A_i\|^2)^{-1}$. By the definition of g_λ, this implies that $g_\lambda \leq \dfrac{1}{1 - \lambda \cdot \max_{i \in \mathcal{M}} \|A_i\|^2}$, which is the desired conclusion for $0 \leq \lambda < (\max_{i \in \mathcal{M}} \|A_i\|^2)^{-1}$. The case when $(\max_{i \in \mathcal{M}} \|A_i\|^2)^{-1} \leq \lambda < \lambda^*$ is trivial. □

The following lemma on power series is proved in [7].

Lemma 3. *Let $\{w_t\}_{t=0,1,\ldots}$ be a sequence of nonnegative scalars satisfying $\sum_{t=0}^{\infty} w_{t+s} \lambda_0^t \leq \beta w_s$, $s = 0, 1, \ldots$, for some constants $\lambda_0 > 0$ and $\beta \geq 1$. Then the power series $\sum_{t=0}^{\infty} w_t \lambda^t$ has its radius of convergence at least $\lambda_1 := \lambda_0/(1 - 1/\beta)$, and*

$$\sum_{t=0}^{\infty} w_t \lambda^t \leq \frac{\beta w_0}{1 - (\beta - 1)(\lambda/\lambda_0 - 1)} < \infty, \quad \forall\ \lambda \in [\lambda_0, \lambda_1).$$

Lemma 3 can be used to prove the following estimate [7].

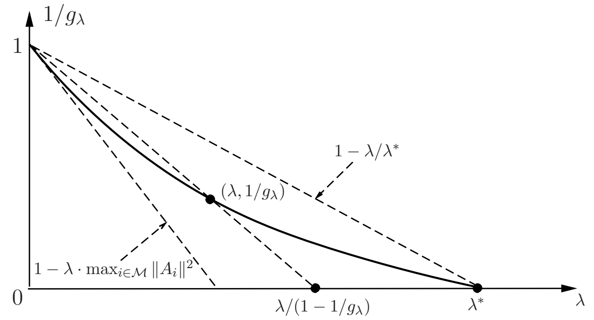

Figure 1: Plot of the function $1/g_\lambda$.

Proposition 3. *$\lambda/(1-1/g_\lambda)$ is non-decreasing for $\lambda \in (0,\lambda^*)$, and bounded by $\frac{\lambda}{1-1/g_\lambda} \le \lambda^*$, $\lambda \in (0,\lambda^*)$.*

As a consequence, we have the following two results.

Corollary 2. *The function $1/g_\lambda$ defined on $[0,\lambda^*)$ is strictly decreasing and Lipschitz continuous with Lipschitz constant $\max_{i\in\mathcal{M}}\|A_i\|^2$. Moreover, $1/g_\lambda \to 0$ as $\lambda \uparrow \lambda^*$.*

Proof. The monotonicity of $1/g_\lambda$ follows directly from that of the function g_λ, as the latter is strictly increasing on $[0,\lambda^*)$. Pick any $\lambda_0, \lambda \in (0,\lambda^*)$ with $\lambda_0 < \lambda$. Then Proposition 3 implies $\lambda/(1-1/g_\lambda) \ge \lambda_0/(1-1/g_{\lambda_0})$; thus

$$\frac{1}{g_\lambda} - \frac{1}{g_{\lambda_0}} \ge -(\lambda-\lambda_0)\frac{1-1/g_{\lambda_0}}{\lambda_0} \ge -(\lambda-\lambda_0)\cdot\max_{i\in\mathcal{M}}\|A_i\|^2,$$

where the last inequality follows from Lemma 2. Similarly at $\lambda_0 = 0$, by Lemma 2, we have $0 \ge 1/g_\lambda - 1/g_{\lambda_0} \ge -(\lambda-\lambda_0)\max_{i\in\mathcal{M}}\|A_i\|^2$ for $\lambda \in [0,\lambda^*)$. This shows that $1/g_\lambda$ is Lipschitz continuous on $[0,\lambda^*)$ with Lipschitz constant $\max_{i\in\mathcal{M}}\|A_i\|^2$. Finally, since by Proposition 3, $0 \le 1/g_\lambda \le 1-\lambda/\lambda^*$ for $\lambda \in (0,\lambda^*)$, letting $\lambda \uparrow \lambda^*$, we have $0 \le \lim_{\lambda\uparrow\lambda^*} 1/g_\lambda \le 0$, i.e., $\lim_{\lambda\uparrow\lambda^*} 1/g_\lambda = 0$. $\square$

Shown in Figure 1 is the plot (in solid line) of a general $1/g_\lambda$ as a function of λ. By the above results, the graph of $1/g_\lambda$ is sandwiched between those of two linear functions: $1-\lambda\max_{i\in\mathcal{M}}\|A_i\|^2$ from the left and $1-\lambda/\lambda^*$ from the right. In addition, given any $\lambda \in (0,\lambda^*)$, the ray (middle dashed line in Figure 1) emitting from the point $(0,1)$ and passing through $(\lambda, 1/g_\lambda)$ intersects the λ-axis at the point $(\frac{\lambda}{1-1/g_\lambda}, 0)$ that moves monotonically to the right towards $(\lambda^*, 0)$ as λ increases by Proposition 3. i.e., the ray rotates around its starting point $(1,0)$ counterclockwise monotonically. In particular, if g_{λ_0} is known for some $\lambda_0 \in [0,\lambda^*)$, then a lower bound of λ^* is given by $\lambda^* \ge \lambda_0/(1-1/g_{\lambda_0})$.

Corollary 3. *The function g_λ for $\lambda \in [0,\lambda^*)$ is continuous, strictly increasing, with $g_\lambda \to +\infty$ as $\lambda \uparrow \lambda^*$. Thus, $G_{\lambda^*}(z)$ has infinite value at some $z \in \mathbb{R}^n$.*

Corollary 3 implies that, as λ increases, λ^* is indeed the first value at which $G_\lambda(\cdot)$ starts to have infinite values.

3.5 Norms Induced by Finite $G_\lambda(z)$

As an immediate result of Proposition 1, for $\lambda \in [0,\lambda^*)$, $\sqrt{G_\lambda(z)}$ is finite, sub-additive, and homogeneous of degree one; thus it defines a norm on the vector space $\mathbb{R}^n$:

$$\|z\|_{G_\lambda} := \sqrt{G_\lambda(z)}, \quad \forall z \in \mathbb{R}^n.$$

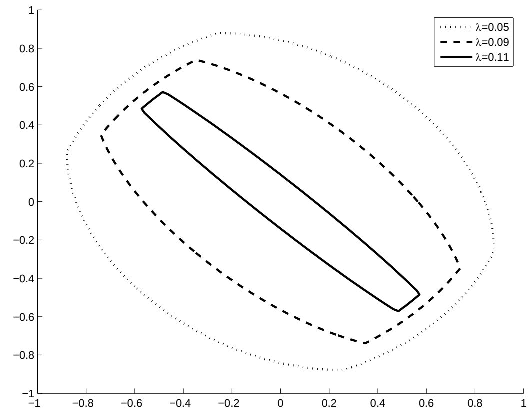

Figure 2: Unit balls of $\|\cdot\|_{G_\lambda}$ for the SLS (4).

This family of norms $\|\cdot\|_{G_\lambda}$ is increasing in λ, or equivalently, the corresponding unit balls shrink as λ increases. See Fig. 2 for the plots of such unit balls for the SLS (4) in Example 1.

The norm $\|\cdot\|_{G_\lambda}$ induces a matrix norm as: $\|A\|_{G_\lambda} := \sup_{z\neq 0}\|Az\|_{G_\lambda}/\|z\|_{G_\lambda}$, $\forall A \in \mathbb{R}^{n\times n}$.

Proposition 4. *Let $\|\cdot\|_{G_\lambda}$ be the norm defined above with respect to the SLS (1) for $\lambda \in [0,\lambda^*)$. Then,*

$$\max_{i\in\mathcal{M}}\|A_i\|_{G_\lambda} = \sqrt{\frac{d_\lambda}{1+\lambda d_\lambda}}, \tag{6}$$

where $d_\lambda := \sup_{\|z\|=1,\, i\in\mathcal{M}} G_\lambda(A_i z)$, $0 \le \lambda < \lambda^*$.

Proof. For any $i \in \mathcal{M}$, by the Bellman equation,

$$\begin{aligned}\|A_i\|^2_{G_\lambda} &= \sup_{z\neq 0}\frac{\|A_i z\|^2_{G_\lambda}}{\|z\|^2_{G_\lambda}} = \sup_{z\neq 0}\frac{G_\lambda(A_i z)}{G_\lambda(z)}\\ &= \sup_{z\neq 0}\frac{G_\lambda(A_i z)}{\|z\|^2+\lambda\cdot\max_{j\in\mathcal{M}} G_\lambda(A_j z)}.\end{aligned}$$

As a result,

$$\begin{aligned}\max_{i\in\mathcal{M}}\|A_i\|^2_{G_\lambda} &= \max_{i\in\mathcal{M}}\sup_{z\neq 0}\frac{G_\lambda(A_i z)}{\|z\|^2+\lambda\cdot\max_{j\in\mathcal{M}} G_\lambda(A_j z)}\\ &= \sup_{z\neq 0}\frac{\max_{i\in\mathcal{M}} G_\lambda(A_i z)}{\|z\|^2+\lambda\cdot\max_{j\in\mathcal{M}} G_\lambda(A_j z)}\\ &= \sup_{z\neq 0}\frac{\max_{i\in\mathcal{M}} G_\lambda(A_i z/\|z\|)}{1+\lambda\cdot\max_{i\in\mathcal{M}} G_\lambda(A_i z/\|z\|)}\\ &= \sup_{\|z\|=1}\frac{\max_{i\in\mathcal{M}} G_\lambda(A_i z)}{1+\lambda\cdot\max_{i\in\mathcal{M}} G_\lambda(A_i z)}\\ &= \frac{\sup_{\|z\|=1,\, i\in\mathcal{M}} G_\lambda(A_i z)}{1+\lambda\cdot\sup_{\|z\|=1,\, i\in\mathcal{M}} G_\lambda(A_i z)}.\end{aligned}$$

In the last step, we have used the fact that $x/(1+\lambda x)$ is continuous and increasing in x for any $\lambda \ge 0$. $\square$

Lemma 4. *As $\lambda \uparrow \lambda^*$, d_λ increases to $+\infty$.*

Proof. Obviously d_λ is non-decreasing in λ. By Corollary 3, g_λ increases to ∞ as $\lambda \uparrow \lambda^*$. Thus given any $M > 0$, there is a $\delta > 0$ such that $g_\lambda > M$ for all $\lambda \in (\lambda^*-\delta, \lambda^*)$, i.e., $G_\lambda(z) > M$ for some $z \in \mathbb{S}^{n-1}$. By the Bellman equation, $G_\lambda(z) = 1 + \lambda\cdot\max_{i\in\mathcal{M}} G_\lambda(A_i z) \ge M$, thus,

$$\max_{i\in\mathcal{M}} G_\lambda(A_i z) \ge (M-1)/\lambda \;\Rightarrow\; d_\lambda \ge \frac{M-1}{\lambda} \ge \frac{M-1}{\lambda^*}.$$

As $\lambda \uparrow \lambda^*$, $M \to +\infty$; thus $d_\lambda \to +\infty$ as desired. □

By (6) and Lemma 4, $\lim_{\lambda\uparrow\lambda^*} \max_{i\in\mathcal{M}} \|A_i\|_{G_\lambda} = (\lambda^*)^{-1/2}$. Hence, for any small $\varepsilon > 0$, we can find a $\lambda < \lambda^*$ such that $\max_{i\in\mathcal{M}} \|A_i\|_{G_\lambda} \le (\lambda^*)^{-1/2} + \varepsilon$, i.e.,

$$\|A_i\|_{G_\lambda} \le (\lambda^*)^{-1/2} + \varepsilon, \quad i \in \mathcal{M}. \tag{7}$$

Note that this particular norm $\|\cdot\|_{G_\lambda}$ is equivalent to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$: $\|z\| \le \|z\|_{G_\lambda} \le \sqrt{g_\lambda}\|z\|$, $\forall z$.

For any trajectory $x(t; z, \sigma)$ of the SLS, we then have

$$\begin{aligned}\|x(t,z,\sigma)\| &\le \|x(t; z, \sigma)\|_{G_\lambda} = \|A_{\sigma(t-1)} x(t-1; z, \sigma)\|_{G_\lambda}\\ &\le \|A_{\sigma(t-1)}\|_{G_\lambda} \cdot \|x(t-1; z, \sigma)\|_{G_\lambda}\\ &\le \left[(\lambda^*)^{-1/2} + \varepsilon\right] \cdot \|x(t-1; z, \sigma)\|_{G_\lambda}\\ &\le \left[(\lambda^*)^{-1/2} + \varepsilon\right]^t \|z\|_{G_\lambda} \le \sqrt{g_\lambda}\left[(\lambda^*)^{-1/2} + \varepsilon\right]^t \|z\|,\end{aligned}$$

for $t = 0, 1, \ldots$. This yields an upperbound on the exponential growth rate of $\|x(t; z, \sigma)\|$ uniformly over the initial state z and the switching sequence σ.

Theorem 2. *Let λ^* be the radius of strong convergence of the SLS. For any $\varepsilon > 0$, we can find $\lambda < \lambda^*$ satisfying (7). Then for any trajectory $x(t; z, \sigma)$ of the SLS,*

$$\|x(t; z, \sigma)\| \le \sqrt{g_\lambda}\left[(\lambda^*)^{-1/2} + \varepsilon\right]^t \|z\|.$$

Remark 1. *It can be shown that $\lambda^* = 1/(\rho^*)^2$, where ρ^* is the joint spectral radius of $\{A_i\}_{i\in\mathcal{M}}$ defined as: $\rho^* := \lim_{k\to\infty} \sup\left\{\|A_{i_1}\cdots A_{i_k}\|^{1/k},\, i_1, \ldots, i_k \in \mathcal{M}\right\}$. Thus Theorem 2 can be viewed as a counterpart of [16, Prop. 4.17] derived using the generating functions. As $\lambda \uparrow \lambda^*$, the norm $\|\cdot\|_{G_\lambda}$ approaches extreme norms of $\{A_i\}_{i\in\mathcal{M}}$ ([1, 17]).*

3.6 Algorithms for Computing $G_\lambda(z)$

We next present some algorithms for computing the finite strong generating functions and for testing whether a given SLS is exponentially stable under arbitrary switching.

First note that $G_\lambda(z)$ as the value function of an infinite horizon optimal control problem is the limit of the value functions of a sequence of finite horizon problems with increasing time horizon. Specifically, define for $k = 0, 1, \ldots$

$$G_\lambda^k(z) := \max_\sigma \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma)\|^2, \quad z \in \mathbb{R}^n. \tag{8}$$

Here, maximum is used instead of supremum as only the first k steps of σ affect the summation.

The above defined functions $G_\lambda^k(z)$ can be computed recursively by: $G_\lambda^0(z) = \|z\|^2$; and for $k = 1, 2, \ldots$,

$$G_\lambda^k(z) = \|z\|^2 + \lambda \cdot \max_{i\in\mathcal{M}} G_\lambda^{k-1}(A_i z), \quad \forall z \in \mathbb{R}^n. \tag{9}$$

Proposition 5. *The sequence of functions $G_\lambda^k(z)$ has the following properties. Let $\lambda \ge 0$ be arbitrary.*

1. *(Monotonicity): $G_\lambda^0(z) \le G_\lambda^1(z) \le G_\lambda^2(z) \le \cdots$.*
2. *(Convergence): $\lim_{k\to\infty} G_\lambda^k(z) = G_\lambda(z)$ for each z.*
3. *(Convexity and Sub-Additivity): For $k = 0, 1, \ldots$, both $G_\lambda^k(z)$ and $\sqrt{G_\lambda^k(z)}$ are convex and locally Lipschitz continuous functions on $\mathbb{R}^n$. Moreover, $\forall z_1, z_2 \in \mathbb{R}^n$,*

$$\sqrt{G_\lambda^k(z_1{+}_2)} \le \sqrt{G_\lambda^k(z_1)} + \sqrt{G_\lambda^k(z_2)}.$$

Proof. For each $z \in \mathbb{R}^n$, let σ_k be a switching sequence achieving the maximum in (8). Then,

$$\begin{aligned}G_\lambda^k(z) &= \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma_k)\|^2 \le \sum_{t=0}^{k+1} \lambda^t \|x(t; z, \sigma_k)\|^2\\ &\le \max_\sigma \sum_{t=0}^{k+1} \lambda^t \|x(t; z, \sigma)\|^2 = G_\lambda^{k+1}(z).\end{aligned}$$

Similarly, we can show $G_\lambda^k(z) \le G_\lambda(z)$. Therefore, we have $\lim_{k\to\infty} G_\lambda^k(z) \le G_\lambda(z)$. The other direction of the inequality can be proved by taking a trajectory $x(t; z, \sigma)$ that achieves the supremum in the definition (2) of $G_\lambda(z)$ and truncating it over a sufficiently large time horizon $[0, k]$. The proof of Property 3 is entirely similar to that of Proposition 1, hence is omitted. In particular, unlike the $G_\lambda(z)$ case, there is no constraint on λ for the continuity of $G_\lambda^k(z)$ due to the finite summation in (8). □

By the above proposition, approximations of the finite strong generating function $G_\lambda(z)$ are provided by $G_\lambda^k(z)$ for k large enough, obtained through the iteration procedure (9). However, for numerical implementation, several issues need to be resolved. First, numerical representations of $G_\lambda^k(z)$ have to be found. For instance, one can represent $G_\lambda^k(z)$ by its values on sufficiently fine grid points of the unit sphere $\mathbb{S}^{n-1}$, with each grid point corresponding to a ray by the homogeneity of $G_\lambda^k(z)$. Second, one needs to estimate the values of $G_\lambda^k(A_i z)$ at those $A_i z$ not aligned with the grid points in order to carry out the recursion (9). By writing such $A_i z$ as a linear combination of nearby grid points and using the convexity of the function $\sqrt{G_\lambda^k(z)}$, an upper estimate of $G_\lambda^k(A_i z)$ can be obtained.

Algorithm 1 Computing $G_\lambda(z)$ on Grid Points of $\mathbb{S}^{n-1}$

Let $\mathcal{S} = \{z_j\}_{j=1}^N$ be a set of grid points of $\mathbb{S}^{n-1}$;
Initialize $k := 0$, and $\widehat{G}_\lambda^0(z_j) := 1$ for all $z_j \in \mathcal{S}$;
repeat
  $k := k + 1$;
  for each $z_j \in \mathcal{S}$ **do**
    for each $i \in \mathcal{M}$ **do**
      Find a minimal subset $\mathcal{S}_{ij}$ of $\mathcal{S}$ whose elements span a convex cone containing $A_i z_j$;
      Write $A_i z_j = \sum_{z_\ell\in\mathcal{S}_{ij}} \alpha_\ell^{ij} z_\ell$ with $\alpha_\ell^{ij} > 0$;
      Compute $g_{ij} := \sum_{z_\ell\in\mathcal{S}_{ij}} \alpha_\ell^{ij} \sqrt{\widehat{G}_\lambda^{k-1}(z_\ell)}$;
    end for
    Set $\widehat{G}_\lambda^k(z_j) := 1 + \lambda \cdot \max_{i\in\mathcal{M}} g_{ij}^2$;
  end for
until k is large enough
return $\widehat{G}_\lambda^k(\cdot)$

The above idea is summarized in Algorithm 1, which is ideally suited for those SLSs with a large number of subsystems and a low dimensional state space, as its computational complexity increases linearly with the number m of subsystems but exponentially with state space dimension n. A similar ray gridding method was used in [18] for computing polyhedral Lyapunov functions of switched linear systems.

After the completion of Algorithm 1, a sequence of mappings $\widehat{G}_\lambda^k : \mathcal{S} \to \mathbb{R}_+$, $k = 0, 1, \ldots$, will be generated. Using

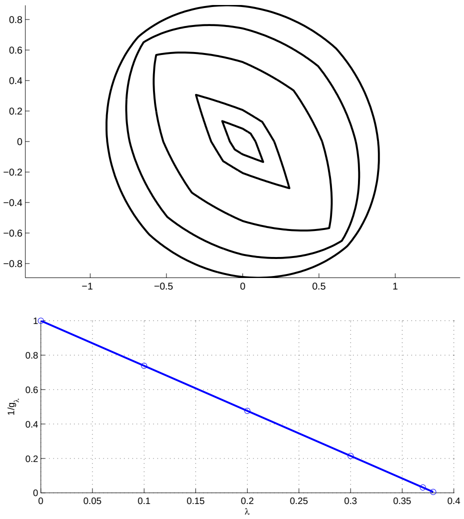

Figure 3: Top: Unit balls of $\|\cdot\|_\lambda$ computed by Algorithm 1 for $\lambda = 0.1, 0.2, 0.3, 0.37, 0.38$ (inward). Bottom: plot of $1/g_\lambda$ for the above values of λ.

the convexity of $\sqrt{G^k_\lambda(z)}$, we can easily show (see [7]) that they provide upperbounds for $G^k_\lambda(z)$ on $\mathcal{S}$.

Proposition 6. $G^k_\lambda(z_j) \leq \widehat{G}^k_\lambda(z_j)$, $\forall z_j \in \mathcal{S}$, $\forall k = 0, 1, \ldots$

By Theorem 1, we have the following stability test.

Corollary 4. *A sufficient condition for the SLS to be exponentially stable under arbitrary switching is that the sequence of mappings $\widehat{G}^k_1 : \mathcal{S} \to \mathbb{R}_+$ obtained by Algorithm 1 is uniformly bounded for all k at each grid point in $\mathcal{S}$.*

Finally, we remark that by repeatedly applying Algorithm 1 to a sequence of λ whose values increase according to the update rule $\lambda_{next} = \lambda/(1 - 1/g_\lambda)$ (see Figure 1), we can obtain underestimates of λ^* with any precision as permitted by the numerical computation errors.

Example 2. *The following example is taken from [2]:*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Algorithm 1 is used to compute the functions $G_\lambda(z)$ of this SLS for different values of λ: λ =0.1, 0.2, 0.3, 0.37, and 0.38. The results are shown in Fig. 3, where the unit balls of the norm $\|\cdot\|_\lambda$ for various λ are plotted in the top figure. The estimated $1/g_\lambda$ for different λ is plotted in the bottom figure which, interestingly, has no discernible difference from a linear function. This phenomenon may partly explain why the joint spectral radius in this case can be obtained analytically as $\rho^* = \frac{1+\sqrt{5}}{2}$ [16]; thus $\lambda^* = 1/(\rho^*)^2 \simeq 0.3820$.

Example 3. *Consider the following SLS in $\mathbb{R}^3$:*

$$A_1 = \begin{bmatrix} 0.5 & 0 & -0.7 \\ 0 & 0.3 & 0 \\ 0 & -0.4 & -0.6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.4 & 0.2 & 0.3 \\ 0 & 0 & 0.3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0.9 & 0.2 & 0.3 \\ -0.2 & 0.3 & -0.5 \end{bmatrix}.$$

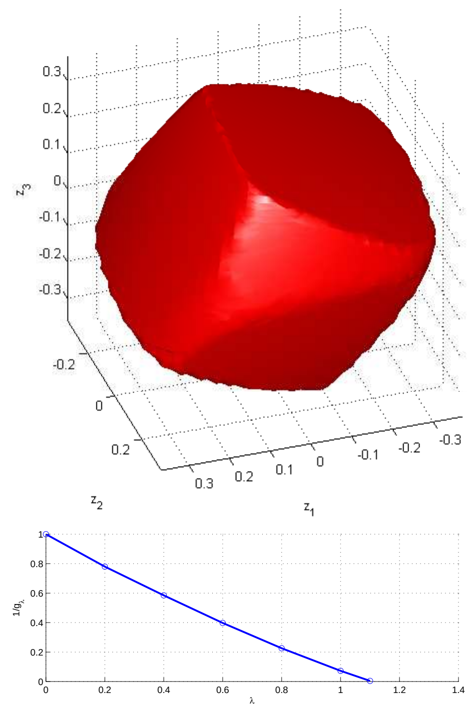

Figure 4: Top: Unit ball of $\|\cdot\|_1$ for the SLS in Example 3. Bottom: $1/g_\lambda$ as a function of λ.

Using Algorithm 1 on 75^2 grid points of the unit sphere $\mathbb{S}^2$, over-estimates of the function $G_\lambda(z)$ are computed. The unit ball corresponding to the estimated norm $\sqrt{G_1(z)}$ is shown in the top of Fig. 4; and the bottom plots the computed $1/g_\lambda$ for $\lambda = 0.2$, 0.4, 0.6, 0.8, 1.1. Since g_λ at $\lambda = 1.1$ is finite, we must have $\lambda^* > 1.1$, hence the given SLS is exponentially stable under arbitrary switching. Indeed, an extrapolation of the function $1/g_\lambda$ assumes zero value at around 1.1064, providing an estimate of the λ^*.

4. WEAK GENERATING FUNCTIONS

4.1 Definition and Properties

For each $z \in \mathbb{R}^n$, the weak generating function $H(\cdot, z) : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is defined as

$$H(\lambda, z) := \inf_\sigma \sum_{t=0}^\infty \lambda^t \|x(t; z, \sigma)\|^2, \quad \forall \lambda \geq 0, \qquad (10)$$

where the infimum is over all switching sequences σ of the SLS. Then $H(\lambda, z)$ is monotonically increasing in λ, with $H(0, z) = \|z\|^2$ when $\lambda = 0$. The threshold

$$\lambda_*(z) := \sup\{\lambda \mid H(\lambda, z) < +\infty\}$$

is called the *radius of weak convergence* of the SLS at z.

For each $\lambda \geq 0$, define the function $H_\lambda : \mathbb{R}^n \to \mathbb{R}_+$ as

$$H_\lambda(z) := H(\lambda, z), \quad \forall z \in \mathbb{R}^n. \qquad (11)$$

which is homogeneous of degree two, and $H_0(z) = \|z\|^2$.

Some properties of the function $H_\lambda(z)$ are listed below. It is noted that many properties of the strong generating function $G_\lambda(z)$ do not have their counterparts for $H_\lambda(z)$.

PROPOSITION 7. *$H_\lambda(z)$ has the following properties.*

1. *(Bellman Equation): For all $\lambda \geq 0$ and all $z \in \mathbb{R}^n$, we have $H_\lambda(z) = \|z\|^2 + \lambda \cdot \min_{i\in\mathcal{M}} H_\lambda(A_i z)$,*
2. *(Invariant Subset): For each $\lambda \geq 0$, the set $\mathcal{H}_\lambda := \{z \in \mathbb{R}^n \mid H_\lambda(z) = \infty\}$ is a subset of $\mathbb{R}^n$ invariant under $\{A_i\}_{i\in\mathcal{M}}$, i.e., $A_i\mathcal{H}_\lambda \subset \mathcal{H}_\lambda$ for all $i \in \mathcal{M}$.*
3. *For $0 \leq \lambda < \frac{1}{\min_{i\in\mathcal{M}} \|A_i\|^2}$, $\|z\|^2 \leq H_\lambda(z) \leq c\|z\|^2$ for some finite constant c.*

PROOF. Property 1 can be obtained by applying the dynamic programming principle to the optimal control problem for minimizing the cost function $\sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^2$. Property 2 is an immediate result of Property 1. For Property 3, one simply note that, by choosing σ to be the sequence σ_0 consisting of a single mode i_0 (thus with no switching) where $i_0 = \text{argmin}_{i\in\mathcal{M}}\|A_i\|$, we have $\|x(t; z, \sigma_0)\|^2 \leq (\min_{i\in\mathcal{M}} \|A_i\|^2)^t\|z\|^2$ for all $t = 0, 1, \ldots$. □

Note that $\mathcal{H}_\lambda$, unlike $\mathcal{G}_\lambda$, is not a subspace of $\mathbb{R}^n$ as it does not contain the origin. In general, $\mathcal{H}_\lambda$ is the union of a (possibly infinite) number of rays with the origin excluded.

4.2 Radius of Weak Convergence

DEFINITION 3. *The radius of weak convergence for the SLS (1), denoted by $\lambda_* \in (0, \infty]$, is defined as*

$$\lambda_* := \sup\{\lambda \mid \textit{there exists a finite constant } c \textit{ such that } H_\lambda(z) \leq c\|z\|^2,\ \forall\, z \in \mathbb{R}^n\}.$$

By Proposition 7, we must have $\lambda_* \geq \frac{1}{\min_{i\in\mathcal{M}} \|A_i\|^2}$. The value of λ_* could reach $+\infty$ if, for instance, starting from any z, there exists at least a switching sequence σ such that the resulting solution $x(t; z, \sigma)$ will reach the origin in at most a finite time T independent of z.

The radius of weak convergence λ_* plays the same role in the stability analysis of the SLS under proper switching as λ^* plays in the stability under arbitrary switching.

THEOREM 3. *The SLS is exponentially stable under proper switching if and only if its radius of weak convergence $\lambda_* > 1$.*

PROOF. Suppose the SLS is exponentially stable under proper switching, i.e., there exist constants $\kappa \geq 1$, $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, there exists a switching sequence σ_z such that $\|x(t; z, \sigma_z)\| \leq \kappa r^t\|z\|$, $t = 0, 1, \ldots$. Then,

$$H_\lambda(z) \leq \sum_{t=0}^{\infty} \lambda^t\|x(t; z, \sigma_z)\|^2 \leq \frac{\kappa^2}{1-\lambda r^2}\|z\|^2, \quad \forall z \in \mathbb{R}^n,$$

for all $\lambda < r^{-2}$. In other words, $\lambda_* \geq r^{-2} > 1$.

Conversely, assume $\lambda_* > 1$. Then $H_1(z)$ satisfies $\|z\|^2 \leq H_1(z) \leq c\|z\|^2$ for some finite constant c. Starting from any $z \in \mathbb{R}^n$, let σ_z be the (possibly multi-valued) state feedback switching policy determined by

$$\sigma_z := \text{argmin}_{i\in\mathcal{M}} H_1(A_i z).$$

Denote by $x(t; z, \sigma_z)$ a solution of the SLS under this state feedback switching policy. We claim that $H_1(z)$ is a Lyapunov function for $x(t; z, \sigma_z)$. Indeed, at time $t = 0$, $z' := x(1, z, \sigma(z))$ satisfies, by the Bellman equation,

$$H_1(z) = \|z\|^2 + \min_{i\in\mathcal{M}} H_1(A_i z) = \|z\|^2 + H_1(z').$$

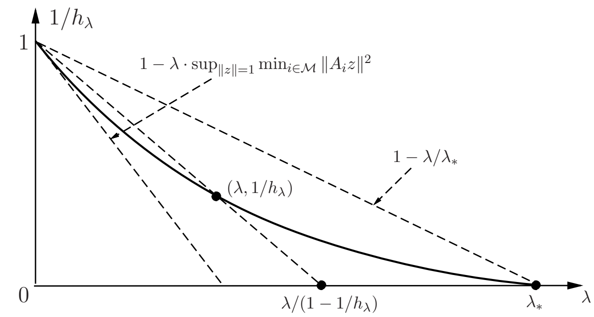

Figure 5: Plot of the function $1/h_\lambda$.

In other words, $H_1(z) - H_1(z') = \|z\|^2$. Similarly,

$$H_1(x(t; z, \sigma_z)) - H_1(x(t + 1; z, \sigma_z)) = \|x(t; z, \sigma_z)\|^2,$$

for all $t = 0, 1, \ldots$. This, together with the fact that $\|z\|^2 \leq H_1(z) \leq c\|z\|^2$, implies that $H_1(z)$ is a Lyapunov function for the closed-loop solutions under the switching policy σ_z. Thus, the SLS is exponentially stable under σ_z. □

Similar to Corollary 1, we can prove the following result.

COROLLARY 5. *Given a SLS with its radius of weak convergence λ_*, for any $r > (\lambda_*)^{-1/2}$, there is a constant κ_r such that starting from each $z \in \mathbb{R}^n$, $\|x(t; z, \sigma_z)\| \leq \kappa_r r^t\|z\|$, $t = 0, 1, \ldots$, for at least some switching sequence σ_z.*

4.3 Quadratic Bounds of Finite $H_\lambda(z)$

For each $\lambda \in [0, \lambda_*)$, define the constant

$$h_\lambda := \sup_{\|z\|=1} H_\lambda(z). \tag{12}$$

Then h_λ is the smallest constant c such that $H_\lambda(z) \leq c\|z\|^2$ for all z. Obviously, h_λ is nondecreasing in λ, and assumes the value of 1 at $\lambda = 0$.

Similar to Proposition 3 for g_λ, using Lemma 3, the following estimate of h_λ is proved in [7].

PROPOSITION 8. *The function $\lambda/(1-1/h_\lambda)$ is nondecreasing for $\lambda \in (0, \lambda_*)$, and bounded by*

$$\frac{\lambda}{1 - 1/h_\lambda} \leq \lambda_*, \quad \forall\lambda \in (0, \lambda_*). \tag{13}$$

The following result follows directly from Proposition 8.

COROLLARY 6. *For each $\lambda \in [0, \lambda_*)$, $1/h_\lambda \leq 1 - \lambda/\lambda_*$. As a result, $1/h_\lambda \to 0$ and $h_\lambda \to \infty$ as $\lambda \uparrow \lambda_*$,*

A counterpart to Lemma 2 for g_λ is proved in [7]:

COROLLARY 7. *For any $\lambda \in [0, \lambda_*)$, we have $1/h_\lambda \geq 1 - \lambda \cdot \sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2$.*

Similar to the proof of Corollary 2, Proposition 8 and Corollary 7 lead to the following result.

COROLLARY 8. *$1/h_\lambda$ defined on $[0, \lambda_*)$ is strictly decreasing and Lipschitz continuous with a Lipschitz constant given by $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2$. As a result, the function h_λ is strictly increasing and locally Lipschitz continuous on $[0, \lambda_*)$.*

A general plot of the function $1/h_\lambda$ for $\lambda \in [0, \lambda_*)$ is shown in Figure 5. The function decreases strictly from 1 at $\lambda = 0$ to 0 as $\lambda \uparrow \lambda_*$. Its graph is sandwiched by those of two linear functions: $1 - \lambda/\lambda_*$ from the right, and $1 - \lambda \cdot \sup_{\|z\|=1} \min_{i \in \mathcal{M}} \|A_i z\|^2$ from the left. Moreover, as λ increases from 0 towards λ_*, the ray emitting from the point $(1, 0)$ and passing through the point $(\lambda, 1/h_\lambda)$ rotates counterclockwise monotonically; and intersects the λ-axis at a point whose λ-coordinate, $\lambda/(1 - 1/h_\lambda)$, provides asymptotically tight lower bound of λ_*.

4.4 Approximating Finite $H_\lambda(z)$

For each $\lambda \in [0, \lambda_*)$, the function $H_\lambda(z)$ is finite everywhere on $\mathbb{R}^n$. We next show that it is the limit of a sequence of functions $H_\lambda^k(z)$, $k = 0, 1, \ldots$, defined by

$$H_\lambda^k(z) := \min_\sigma \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma)\|^2, \quad \forall z \in \mathbb{R}^n. \tag{14}$$

As the value functions of an optimal control problem with finite horizon, $H_\lambda^k(z)$ can be computed recursively as follows: $H_\lambda^0(z) = \|z\|^2$, $\forall z \in \mathbb{R}^n$; and for $k = 1, 2, \ldots$,

$$H_\lambda^k(z) = \|z\|^2 + \lambda \cdot \min_{i \in \mathcal{M}} H_\lambda^{k-1}(A_i z), \quad \forall z \in \mathbb{R}^n.$$

Equivalently, we can write

$$H_\lambda^k(z) = \min\{z^T P z : P \in \mathcal{P}_k\}, \quad \forall z \in \mathbb{R}^n, \tag{15}$$

where $\mathcal{P}_k$, $k = 0, 1, \ldots$, is a sequence of sets of positive definite matrices defined by: $\mathcal{P}_0 = \{I\}$; and for $k = 1, 2, \ldots$,

$$\mathcal{P}_k = \{I + \lambda A_i^T P A_i \,|\, P \in \mathcal{P}_{k-1},\, i \in \mathcal{M}\}. \tag{16}$$

PROPOSITION 9. *$H_\lambda^k(z)$ has the following properties.*

1. *(Monotonicity): $H_\lambda^0 \leq H_\lambda^1 \leq H_\lambda^2 \leq \cdots \leq H_\lambda$.*
2. *(Convergence): For $\lambda \in [0, \lambda_*)$, $H_\lambda^k(z)$ converges exponentially fast to $H_\lambda(z)$ as $k \to \infty$: for $k = 0, 1, \ldots$,*

$$|H_\lambda^k(z) - H_\lambda(z)| \leq h_\lambda^2 (1 - 1/h_\lambda)^{k+1} \|z\|^2, \quad \forall z \in \mathbb{R}^n.$$

PROOF. Fix $k \geq 1$. For each $z \in \mathbb{R}^n$, let σ_k be a switching sequence achieving the minimum in (14). Then,

$$H_\lambda^k(z) = \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma_k)\|^2 \geq \sum_{t=0}^{k-1} \lambda^t \|x(t; z, \sigma_k)\|^2 \geq \min_\sigma \sum_{t=0}^{k-1} \lambda^t \|x(t; z, \sigma)\|^2 = H_\lambda^{k-1}(z).$$

Similarly, we have $H_\lambda^k(z) \leq H_\lambda(z)$, proving the monotonicity. Next assume $\lambda \in [0, \lambda_*)$. Then $\|z\|^2 \leq H_\lambda^k(z) \leq H_\lambda(z) \leq h_\lambda \|z\|^2$, $\forall z \in \mathbb{R}^n$, $k = 0, 1, \ldots$. For any $z \in \mathbb{R}^n$ and $k = 0, 1, \ldots,$, let σ_k be a switching sequence so that $\hat{x}(t) := x(t; z, \sigma_k)$ achieves the minimum in (14). For each $s = 0, 1, \ldots, k-1$, since $\hat{x}(t)$ is also optimal over the time horizon $s \leq t \leq k$, we have

$$\begin{aligned} H_\lambda^{k-s}(\hat{x}(s)) &= \sum_{t=0}^{k-s} \lambda^t \|\hat{x}(t+s)\|^2 \\ &= \|\hat{x}(s)\|^2 + \lambda \sum_{t=0}^{k-s-1} \lambda^t \|\hat{x}(t+s+1)\|^2 \\ &= \|\hat{x}(s)\|^2 + \lambda H_\lambda^{k-s-1}(\hat{x}(s+1)). \end{aligned}$$

Note that $\|\hat{x}(s)\|^2 \geq H_\lambda^{k-s}(\hat{x}(s))/h_\lambda$. Therefore, the above equality implies, for $s = 0, \ldots, k-1$,

$$H_\lambda^{k-s-1}(\hat{x}(s+1)) \leq \lambda^{-1}(1 - 1/h_\lambda) H_\lambda^{k-s}(\hat{x}(s)).$$

Applying this inequality for $s = k-1, k-2, \ldots, 0$, we have

$$\begin{aligned} \|\hat{x}(k)\|^2 &= H_\lambda^0(\hat{x}(k)) \leq \lambda^{-1}(1 - 1/h_\lambda) H_\lambda^1(\hat{x}(k-1)) \leq \cdots \\ &\leq \lambda^{-k}(1 - 1/h_\lambda)^k H_\lambda^k(z) \leq \lambda^{-k} h_\lambda (1 - 1/h_\lambda)^k \|z\|^2. \end{aligned}$$

Using this and considering the switching sequence that first follows σ_k for k steps and thereafter follows an infinite-horizon optimal σ_* starting from the state $\hat{x}(k)$, we obtain

$$\begin{aligned} H_\lambda(z) &\leq \sum_{t=0}^{k} \lambda^t \|\hat{x}(t)\|^2 + \sum_{t=k+1}^{\infty} \lambda^t \|x(t-k; \hat{x}(k), \sigma_*)\|^2 \\ &= H_\lambda^k(z) + \lambda^k \left[H_\lambda(\hat{x}(k)) - \|\hat{x}(k)\|^2\right] \\ &\leq H_\lambda^k(z) + (h_\lambda - 1)\lambda^k \|\hat{x}(k)\|^2 \\ &\leq H_\lambda^k(z) + h_\lambda^2 (1 - 1/h_\lambda)^{k+1} \|z\|^2. \end{aligned}$$

As $H_\lambda^k(z) \leq H_\lambda(z)$, this proves the convergence property. □

Thus, $\{H_\lambda^k(z)\}_{k=0,1,\ldots}$ is a sequence of functions continuous in (λ, z) that converges uniformly on $[0, \lambda_0] \times \mathbb{S}^{n-1}$ to $H_\lambda(z)$ for any $\lambda_0 \in [0, \lambda_*)$. Then, $H_\lambda(z)$ is also continuous on $[0, \lambda_0] \times \mathbb{S}^{n-1}$, hence on $[0, \lambda_*) \times \mathbb{R}^n$, by its homogeneity and the arbitrariness of λ_0.

COROLLARY 9. *The function $H_\lambda(z) = H(\lambda, z)$ is continuous in (λ, z) on $[0, \lambda_*) \times \mathbb{R}^n$. As a result, the function h_λ defined in (12) is continuous on $[0, \lambda_*)$.*

4.5 Relaxation Algorithm for Computing $H_\lambda(z)$

By Proposition 9, $H_\lambda^k(z)$ for large k provide increasingly accurate estimates of $H_\lambda(z)$. By (15), to characterize $H_\lambda^k(z)$, it suffices to compute the set $\mathcal{P}_k$. To deal with the rapidly increasing size of $\mathcal{P}_k$ as k increases, we introduce the following complexity reduction technique. A subset $\mathcal{P}_k^\varepsilon \subset \mathcal{P}_k$ is called *ε-equivalent* to $\mathcal{P}_k$ for some $\varepsilon > 0$ if

$$H_\lambda^{k,\varepsilon}(z) := \min_{P \in \mathcal{P}_k^\varepsilon} z^T P z \leq \varepsilon \|z\|^2 + \min_{P \in \mathcal{P}_k} z^T P z, \quad \forall z \in \mathbb{R}^n.$$

A sufficient condition for this to hold is that, $\forall P \in \mathcal{P}_k$, $P + \varepsilon I_n \succ \sum_{Q \in \mathcal{P}_k^\varepsilon} \alpha_Q \cdot Q$ for some constants $\alpha_Q \geq 0$, $\forall Q \in \mathcal{P}_k^\varepsilon$, adding up to 1. This leads to a procedure of removing matrices from $\mathcal{P}_k$ iteratively until a minimal ε-equivalent subset $\mathcal{P}_k^\varepsilon$ is achieved. By applying this procedure at each step of the iteration (16), we obtain Algorithm 2, which yields over approximations of $H_\lambda^k(z)$ for all $k = 0, 1, \ldots$ with uniformly bounded approximation errors according to Proposition 10 below.

Algorithm 2 Computing Over Approximations of $H_\lambda^k(z)$.

Initialize $k := 0$, $\tilde{\mathcal{P}}_0^\varepsilon := \{I_n\}$, and $\mathcal{P}_0^\varepsilon := \tilde{\mathcal{P}}_0^\varepsilon$;
repeat
  $k := k + 1$;
  $\tilde{\mathcal{P}}_k^\varepsilon := \{I + \lambda A_i^T P A_i \,|\, i \in \mathcal{M},\, P \in \mathcal{P}_{k-1}^\varepsilon\}$;
  Find an ε-equivalent subset $\mathcal{P}_k^\varepsilon \subset \tilde{\mathcal{P}}_k^\varepsilon$;
until k is large enough
return $H_\lambda^{k,\varepsilon}(z) := \min\{z^T P z \,|\, P \in \mathcal{P}_k^\varepsilon\}$.

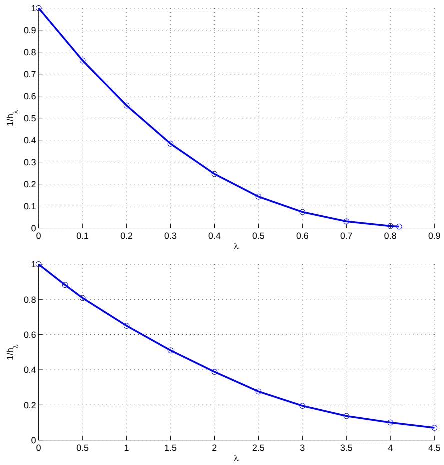

Figure 6: Plots of $1/h_\lambda$ for SLS in Example 2 (top) and Example 3 (bottom).

Proposition 10. *For any* $\lambda \in [0, \lambda_*)$ *and* $k = 0, 1, \ldots,$

$$H_\lambda^k(z) \le H_\lambda^{k,\varepsilon}(z) \le (1+\varepsilon) H_\lambda^k(z). \qquad (17)$$

Proof. Let $\lambda \in [0, \lambda_*)$. We prove (17) by induction. It is true at $k = 0$. Assume it holds for $k-1 \ge 0$. Define

$$\tilde{H}_\lambda^{k,\varepsilon}(z) := \min_{P \in \tilde{\mathcal{P}}_k^\varepsilon} z^T P z = \|z\|^2 + \lambda \min_{i \in \mathcal{M}} H_\lambda^{k-1,\varepsilon}(A_i z).$$

Then, for any $i \in \mathcal{M}$, by the induction hypothesis,

$$\begin{aligned}\tilde{H}_\lambda^{k,\varepsilon}(z) &\le \|z\|^2 + \lambda H_\lambda^{k-1,\varepsilon}(A_i z) \\ &\le \|z\|^2 + \lambda(1+\varepsilon) H_\lambda^{k-1}(A_i z).\end{aligned}$$

Since $\mathcal{P}_k^\varepsilon$ is an ε-equivalent subset of $\tilde{\mathcal{P}}_k^\varepsilon$, we then have

$$H_\lambda^{k,\varepsilon}(z) \le \varepsilon \|z\|^2 + \tilde{H}_\lambda^{k,\varepsilon}(z) \le (1+\varepsilon)\left[\|z\|^2 + \lambda H_\lambda^{k-1}(A_i z)\right].$$

By the Bellman equation, $H_\lambda^k(z) = \|z\|^2 + \lambda H_\lambda^{k-1}(A_i z)$ for some $i \in \mathcal{M}$. Choosing this i in the above inequality implies

$$H_\lambda^{k,\varepsilon}(z) \le (1+\varepsilon) H_\lambda^k(z), \quad \forall z \in \mathbb{R}^n.$$

That $H_\lambda^{k,\varepsilon}(z) \ge H_\lambda^k(z)$ can also be trivially proved. □

Using Algorithm 2, over approximations $H_\lambda^{k,\varepsilon}(z)$ of $H_\lambda^k(z)$ are obtained with arbitrary precisions for large k and small ε. The estimated $1/h_\lambda$ as a function of λ is plotted in Figure 6 for the SLS in Example 2 (top) and Example 3 (bottom), respectively. It can be seen that in both cases, $1/h_\lambda$ decreases from 1 at $\lambda = 0$ to 0 at $\lambda = \lambda_*$ (in the second case, high computational complexity prevents us from getting accurate estimates for λ close to λ_*). Another interesting observation is that in each case $1/h_\lambda$ is roughly a convex function and "more curved" than the plots of $1/g_\lambda$.

5. CONCLUSION

It is found that the strong generating functions characterize the maximum exponential growth rate of the trajectories of switched linear systems, and as a result yield an effective stability test of the systems. Similar results are obtained for the system stability under proper switching using the weak generating functions. The two types of generating functions have many desirable properties that make their efficient numerical computations possible.

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