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# Closed-form solution of P4P or the three-dimensional resection problem in terms of Möbius barycentric coordinates

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**Abstract.** The *perspective 4 point* (P4P) problem - also called the *three-dimensional resection problem* - is solved by means of a new algorithm: At first the unknown Cartesian coordinates of the *perspective center* are computed by means of *Möbius barycentric coordinates*. Secondly these coordinates are represented in terms of observables, namely space angles in the five-dimensional simplex generated by the unknown point and the four known points. Substitution of *Möbius barycentric coordinates* leads to the unknown Cartesian coordinates (2.8)–(2.10) of *Box 2.2.* The unknown distances within the five-dimensional simplex are determined by solving the *Grunert equations*, namely by forward reduction to one algebraic equation (3.8) of order four and backward linear substitution. Tables 1.–4. contain a numerical example. Finally we give a reference to the solution of the 3 point (P3P) problem, the *two-dimensional resection problem*, namely to the *Ansermet barycentric coordinates* initiated by *C.F. Gauß* (1842), *A.Schreiber* (1908) and *A. Ansermet* (1910).

## 0. Introduction

The perspective 4 point (*P4P*) problem - also called the three-dimensional resection problem - is the problem of finding the position and orientation of a sensor (camera, theodolite) with respect to a scene object from 4 correspondence points, a fundamental problem in geodetic positioning, photogrammetry, machine and computer vision. Closed form solutions of the three-dimensional resection problem have been presented by *E. Grafarend et al* (the GLS algorithm, 1989), further developed by *P. Lohse* (1990) and by *E. Grafarend* and *A. Mader* (1991), *E.L. Merritt* (the M algorithm 1949), *M.A. Fischler and R.C. Bolles* (the FB algorithm, 1981), *S. Linnainmaa et al* (the LHD algorithm, 1988), *Z.Q. Zeng and X.B. Wang* (1992), in particular, based upon early works by *J.A. Grunert* (the G algorithm, 1841), *S. Finsterwalder and W. Scheufele* (the FS algorithm, 1903) and *F.J. Müller* (1925). *R.M. Haralick et al* (1994) evaluated six algorithms of type (*i*)G, (*ii*)FS, (*iii*)M, (*iv*)FB, (*v*)LHD and (*vi*) GLS for computer vision applications. Here we develop a new algorithm for *P4P*, the three-dimensional resection problem from four given points, motivated by the implementation of *Möbius barycentric coordinates*. A comparison should be made with the *P4P* solution presented by *R. Horaud et al* (1989) where the term *PnP* is introduced. The *first paragraph* accordingly is devoted to the formulation of P4P in terms of *Möbius barycentric coordinates* paying attention to the introduction of a *local affine basis* and *homogeneous coordinates*, the notion of a *Grassmann manifold* has been found particularly useful. The *P4P* solution in terms of *Möbius barycentric coordinates* is transformed into a form with respect to observables, *in paragraph two*. In contrast, *paragraph three* generates the *P4P* solution in terms of distances and observables of type space angles, namely with respect to the *G algorithm* as an example. A numerical example of the new algorithm is presented in *paragraph four*. In *Appendix A* we pay our tribute to the solution of the *two-dimensional resection problem* of type *P3P* presented by *A. Ansermet* (1910), namely in terms of *Ansermet barycentric coordinates* based upon fundamental works by *C.F. Gauß* (1842) and *A. Schreiber* (1908). Indeed the newly devolped algorithm for solving *P4P*, the three-dimensional resection problem, had been motivated by *the A algorithm of A. Ansermet* (1910) of the two-dimensional resection problem, visualized by *W. Pachelski* (1994). *Appendix B* is an algorithmic solution for the orientation parameters which computes P4P in order to extend in a second phase to the three-dimensional intersection problem, the "forward" pose problem. The closed-form solution of *P4P* as presented in the following supplies us directly with Cartesian coordinates of the perspective center which is resected relative to four given points. The underlying nonlinear adjustment problem needs to be treated in a forthcoming paper.

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## 1. P4P in terms of *Möbius* barycentric coordinates

The primary situation of a P4P in the three-dimensional Euclidean manifold $\mathbf{E}^3 := \{\mathscr{R}^3, g_{\mu\nu}\}$ of standard metric $g_{\mu\nu}$ subject to $\mu, \nu \in \{1,2,3\}$ is as follows: *Directions* at an unknown point $p \in \mathbf{E}^3$ are observed to at least four known points $p_i \in \mathbf{E}^3$ subject to $i, j \in \{1,2,3,4\}$. Equivalently the measurements may be all possible combinational angles between two unit vectors $\overrightarrow{pp_i}/\|\overrightarrow{pp_i}\|$ and $\overrightarrow{pp_j}/\|\overrightarrow{pp_j}\|$, respectively centred at the point $p$. *Figure 1.* illustrates the graph of the five-dimensional simplex. In total there are six possible measurements of space angles defined by $\cos\psi_{ij} := < \overrightarrow{pp_i}|\overrightarrow{pp_j} > /\|\overrightarrow{pp_i}\|/\|\overrightarrow{pp_j}\|$ subject to the inner product $< \overrightarrow{pp_i}|\overrightarrow{pp_j} >$ and the norm $\|\overrightarrow{pp_i}\|$, $\|\overrightarrow{pp_j}\|$ respectively. *If* four known points $p_i \in \mathbf{E}^3$ are given, *then* the P4P is defined by an unknown point $p \in \mathbf{E}^3$ and six angular measurements $\psi_{ij}$. In addition, in geodesy, photogrammetry, machine and computer vision there is the need for the *orientation elements* of the theodolite, the camera or the CCD sensor. Likewise in the case of photogrammetry, where $p$ is the *perspective centre* of the camera or the CCD sensor, the original measurements are *image coordinates* of the point $p_i, p_j \in \mathbf{E}^3$, respectively, $i \neq j$, which are related to space angles $\psi_{ij}$ according to (1.1) of *Box 1.1*. In contrast, in terms of horizontal and vertical directions, respectively, namely $\{\alpha_i, \beta_i\}, \{\alpha_j, \beta_j\}, i \neq j$, measured by a *theodolite*, the space angle $\psi_{ij}$ are represented by (1.2) of *Box 1.2.*

*Box 1.1:* Representation of space angles $\psi_{ij}$ in terms of image coordinates $(\bar{x}_i, \bar{y}_i, \bar{z}_i), (\bar{x}_j, \bar{y}_j, \bar{z}_j)$ of points $p_i$ and $p_j$ with respect to an orthogonal Euclidean frame centred at the *perspective centre* subject to $\bar{z}_i = \bar{z}_j = -f$, where $f$ is the focal length of the *camera* or the *CCD sensor*.

$$\cos\psi_{ij} = \frac{\bar{x}_i\bar{x}_j + \bar{y}_i\bar{y}_j + \bar{z}_i\bar{z}_j}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + \bar{z}_i^2}\sqrt{\bar{x}_j^2 + \bar{y}_j^2 + \bar{z}_j^2}} = \frac{\bar{x}_i\bar{x}_j + \bar{y}_i\bar{y}_j + f^2}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + f^2}\sqrt{\bar{x}_j^2 + \bar{y}_j^2 + f^2}} \quad (1.1)$$

*Box 1.2:* Representation of space angles $\psi_{ij}$ in terms of spherical coordinates $(\alpha_i, \beta_i), (\alpha_j, \beta_j)$ of points $p_i$ and $p_j$ with respect to a *theodolite* orthogonal Euclidean frame ($\alpha_i, \alpha_j$ : horizontal directions, $\beta_i, \beta_j$ : vertical directions).

$$\cos\psi_{ij} = \cos\beta_i \cos\beta_j \cos(\alpha_j - \alpha_i) + \sin\beta_i \sin\beta_j \quad (1.2)$$

In order to generate *Möbius barycentric coordinates* of the points $p_i$ and $p$, respectively, we introduce the *affine basis* (*A.F. Möbius, 1827; H.S.M. Coxeter, 1969*)

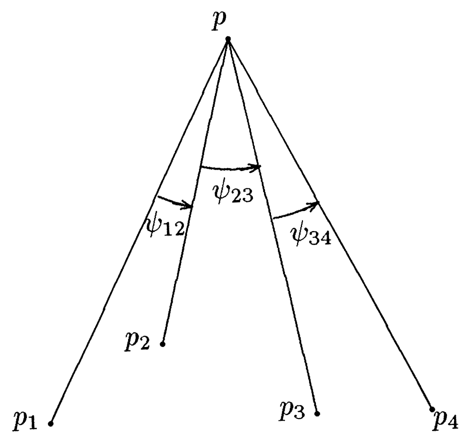


**Fig. 1.** P4P or three-dimensional resection problem

$$\{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \overrightarrow{p_1p_4}\}$$

or *equivalently*

$$\{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_1\}$$

which span an $\mathscr{R}^3$ equipped with a general metric $g_{\mu\nu}$. With respect to the *tetrahedron* $\{p_1, p_2, p_3, p_4\}$, *Figure 2.* is a visualization of the *affine basis* subject to *affine geometry*.

Note that an *affine basis* is defined as a basis of an $\Re^3$ which is *translational invariant* or *equivariant under the action of the translation group*. In addition, in the definition of the affine basis we have used the *equivalence relation* $p_1 \sim \mathbf{x}_1, p_2 \sim \mathbf{x}_2, \quad p_3 \sim \mathbf{x}_3, p_4 \sim \mathbf{x}_4,$ where $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ are *placement vectors.*

Relative to $p_1 \sim \mathbf{x}_1$ *the point* $p \sim \mathbf{x}$ can be represented in the affine basis by (1.3)–(1.7) of *Box 1.3*, where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are the *Möbius barycentric coordinates* of $p \sim \mathbf{x}$. In particular, we have introduced $\lambda_1 := 1 - (\lambda_2 + \lambda_3 + \lambda_4)$. As soon as we cover $\mathbf{E}^3$ by *Cartesian coordinates* $\{x, y, z\}$ in one global chart via (1.6) we are led *from* (1.5) *to* (1.7). $\{x, y, z, 1\}$ are called *homogeneous coordinates* of $p$ for the following reason.

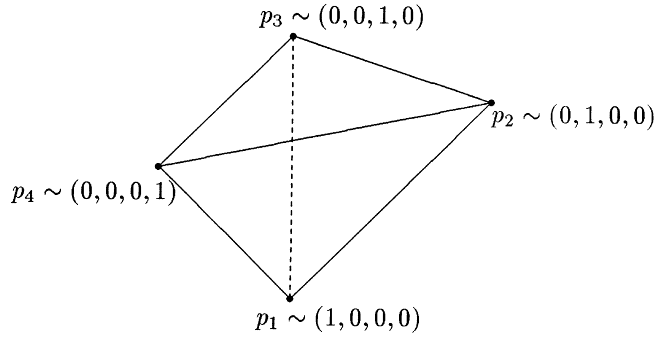


**Fig. 2.** *Möbius barycentric coordinates* of four points, $p_1, p_2, p_3, p_4$, which constitute an affine basis according to (1.11)

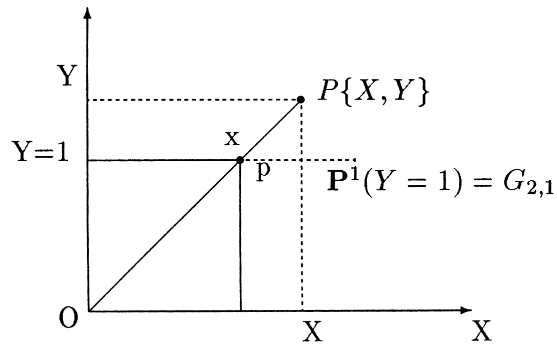


**Fig. 3.** Homogeneous coordinates $\{x, 1\}$, projective geometry, $\{X, Y\} \in \mathscr{R}^2$, rational function $x = X/Y, 1 = Y/Y$

As illustrated by *Figure 3.*, the point $p$ in $\mathscr{R}^4$ is connected to the origin O by a straight line which we refer to as the *projection line* with O as the *perspective centre*. The projection line is supposed to intersect the *three-dimensional Grassmann manifold* $G_{4,3}(X_4 = 1)$, through $X_4 = 1$, a linear *submanifold* of $\mathscr{R}^4$. In order to visualize this *perspective projection* we present two examples (*1.1, 1.2*)

*Box 1.3: Möbius barycentric coordinates*, affine basis.

$$\mathbf{x} - \mathbf{x}_1 = (\mathbf{x}_2 - \mathbf{x}_1)\lambda_2 + (\mathbf{x}_3 - \mathbf{x}_1)\lambda_3 + (\mathbf{x}_4 - \mathbf{x}_1)\lambda_4 \quad (1.3)$$

$$\mathbf{x} = \mathbf{x}_1(1-\lambda_2-\lambda_3-\lambda_4)+\mathbf{x}_2\lambda_2 + \mathbf{x}_3\lambda_3 + \mathbf{x}_4\lambda_4 \quad (1.4)$$

$$\mathbf{x} = \mathbf{x}_1\lambda_1 + \mathbf{x}_2\lambda_2 + \mathbf{x}_3\lambda_3 + \mathbf{x}_4\lambda_4 \quad (1.5)$$

subject to

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \quad (1.6)$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \quad (1.7)$$

*Example 1.1 and 1.2* motivate the representation of the coordinates $\{x, y, z\} \in G_{4,3}(X_4 = 1)$ in terms of the *rational functions*

$$x = \frac{X_1}{X_4} \quad y = \frac{X_2}{X_4} \quad z = \frac{X_3}{X_4} \quad (1.8)$$

subject to $\{X_1, X_2, X_3, X_4\} \in \mathscr{R}^4$.

As being outlined by *Example 1.3 and 1.4* the name "*homogeneous coordinates*" is derived from the fact that any inhomogeneous algebraic manifold in the *Grassmann manifold* $G_{4,3}$ *under the action of the perspective group* (1.8) is transformed into a *homogeneous algebraic manifold* in $\mathscr{R}^4$. Finally in *Box 1.4* we collect some basic properties of *Möbius barycentric coordinates* which will make obvious its name.

*Example 1.1:* Homogeneous coordinates $\{x, 1\}$, projective geometry, $\{X, Y\} \in \mathscr{R}^2$.

According to *Figure 3.* the projection line OP intersects the *Grassmann manifold* $G_{2,1}(Y = 1)$ through $Y = 1$ *or* the line $x \in \mathbf{P}^1$ parallel to $X$ through $Y = 1$, within $\mathscr{R}^2$. Obvious is the *ratio*

$$\frac{x}{X} = \frac{1}{Y} \Longrightarrow x = \frac{X}{Y} \quad \text{(Ex.1.1 : 1)}$$

*Example 1.2:* Homogeneous coordinates $\{x, y, 1\}$, projective geometry, $\{X, Y, Z\} \in \mathscr{R}^3$.

According to *Figure 4.* the projective line OP intersects the *Grassmann manifold* $G_{3,2}(Z = 1)$ through $Z = 1$ *or* the plane $\{x, y\} \in \mathbf{P}^2$ parallel to $\{X, Y\}$ through $Z = 1$, within $\mathscr{R}^3$. Here we want to prove the *rational functions*

$$x = \frac{X}{Z} \quad y = \frac{Y}{Z} \quad \text{(Ex.1.2 : 1)}$$

Firstly in the plane $\{x, y\} \in \mathbf{P}^2$ we introduce *polar coordinates* in form of

$$x = r\cos\alpha \quad y = r\sin\alpha \quad \text{(Ex.1.2 : 2)}$$

*Secondly* we refer to ratios

$$\frac{x}{R} = \frac{\sqrt{x^2+y^2}}{\sqrt{X^2+Y^2}} = \frac{1}{Z} \Longrightarrow$$

$$r = \frac{\sqrt{X^2+Y^2}}{Z} \quad \text{(Ex.1.2 : 3)}$$

$$\tan\alpha = \frac{y}{x} = \frac{Y}{X} \Longrightarrow$$

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{Y}{\sqrt{X^2+Y^2}}$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{X}{\sqrt{X^2+Y^2}}$$

$$\Longrightarrow \quad x = r\cos\alpha = \frac{X}{Z} \quad y = r\sin\alpha = \frac{Y}{Z} \quad \text{(Ex.1.2 : 4)}$$

*Example 1.3:* Homogeneous coordinates $\{x, y, 1\}$, projective geometry, *Grassmann manifold*, $G_{3,2}, (Z = 1)$.

Assume an *ellipse* $\mathbf{E}^1_{a,b}$ to be given in the *Grassmann manifold* $G_{3,2}(Z = 1)$, namely in the plane $\{x, y\} \in \mathscr{R}^2$ by means of (Ex 1.3:1). Under the central projection outlined in *Example 1.2*, namely by means

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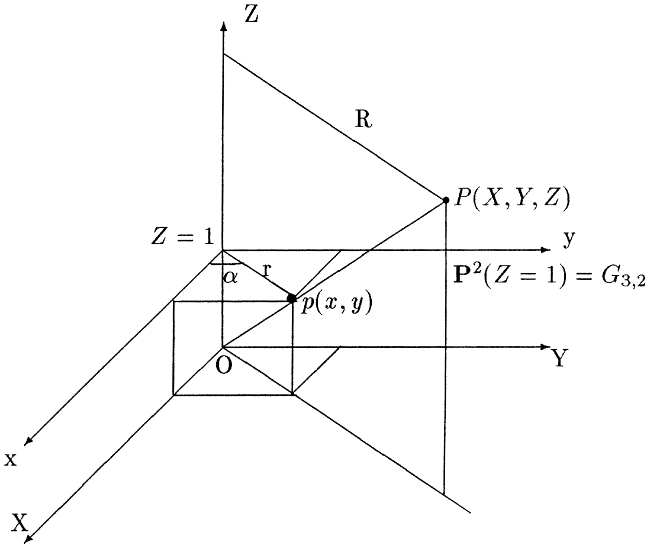


**Fig. 4.** Homogeneous coordinates $\{x, y, 1\}$, projective geometry, $\{X, Y, Z\} \in \mathscr{R}^3$, rational function $x = X/Z, y = Y/Z, 1 = Z/Z$

(*Example 1.3 Continued*)

of the rational functions (Ex 1.2:1) the *ellipse* $\mathbf{E}^1_{a,b}$ as an *inhomogeneous algebraic manifold* is mapped into the cone $\boldsymbol{C}^2_{a,b}$ as a *homogeneous algebraic manifold* (Ex. 1.3:2)

$$\mathbf{E}^1_{a,b} := \{\mathrm{x} \in \mathscr{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b \in \mathscr{R}^+, a > b\}$$
$$= \{\mathrm{x} \in \mathscr{R}^2 | b^2x^2 + a^2y^2 - a^2b^2 = 0, \quad a, b \in \mathscr{R}^+, a > b\} \qquad \text{(Ex.1.3 : 1)}$$

$$x = \frac{X}{Z} \quad y = \frac{Y}{Z} \qquad \text{(Ex.1.2 : 1)}$$

$$\mathbf{C}^2_{a,b} := \{\mathbf{X} \in \mathscr{R}^3 | b^2\frac{X^2}{Z^2} + a^2\frac{Y^2}{Z^2} - a^2b^2 = 0\}$$
$$= \{\mathbf{X} \in \mathscr{R}^3 | b^2X^2 + a^2Y^2 - a^2b^2Z^2 = 0\} \qquad \text{(Ex.1.3 : 2)}$$

*Example 1.4:* Homogeneous coordinates $\{x, y, z, 1\}$, projective geometry, *Grassmann manifold* $G_{4,3}$ $(X_4 = 1)$.

Assume the triaxial ellipsoid $\mathbf{E}^2_{a,b,c}$ to be given in the *Grassmann manifold* $G_{4,3}(X_4 = 1)$, namely in the hyperplane $\{x, y, z\} \in \mathbf{P}^3$, a three-dimensional linear submanifold of $\mathscr{R}^4$ by means of (Ex. 1.4:1). Under the central projection outlined by (1.8) as *rational*

*Continued*

(*Example 1.4 Continued*)

function the triaxial ellipsoid $\mathbf{E}^2_{a,b,c}$ as an *inhomogeneous algebraic manifold* is mapped into the *hypercone* $\mathbf{C}^3_{a,b,c}$ as a homogeneous algebraic manifold (*Ex.* 1.4:2).

$$\mathbf{E}^2_{a,b,c} := \{x \in \mathscr{R}^3 | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a, b, c \in \mathscr{R}^+, a > b > c\}$$
$$= \{x \in \mathscr{R}^3 | b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0, \quad a, b, c \in \mathscr{R}^+, a > b > c\} \qquad \text{(Ex.1.4 : 1)}$$

$$x = \frac{X_1}{X_4} \quad y = \frac{X_2}{X_4} \quad z = \frac{X_3}{X_4} \qquad (1.8)$$

$$\mathbf{C}^3_{a,b,c} := \{\mathbf{X} \in \mathscr{R}^4 | \quad b^2c^2\frac{X_1^2}{X_4^2} + c^2a^2\frac{X_2^2}{X_4^2} + a^2b^2\frac{X_3^2}{X_4^2} - a^2b^2c^2 = 0\}$$
$$= \{\mathbf{X} \in \mathscr{R}^4 | \quad b^2c^2X_1^2 + c^2a^2X_2^2 + a^2b^2X_3^2 - a^2b^2c^2X_4^2 = 0\} \qquad \text{(Ex.1.4 : 2)}$$

*Box 1.4:* ;Properties of barycentric coordinates.

*If* the point $p \sim \mathbf{x}$ is located in the centre of the tetrahedron $\{p_1, p_2, p_3, p_4\} \sim \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, namely

$$X = (x_1 + x_2 + x_3 + x_4)/4$$
$$Y = (y_1 + y_2 + y_3 + y_4)/4 \qquad (1.9)$$
$$Z = (z_1 + z_2 + z_3 + z_4)/4$$

*then* the barycentric coordinates

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (1.10)$$

namely $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{1/4, 1/4, 1/4, 1/4\}$ results. This property of barycentric coordinates made *Möbius* (1827) to refer them to a "*barycentre*". Since the barycentric coordinates sum up to one, they form a partition of unit.

*If* the point $p \sim \mathbf{x}$ is located on one of the points of the tetrahedron, say $p = p_1 \sim \mathbf{x} = \mathbf{x}_1$, *then* the barycentric coordinates

*Continued*

(*Box 1.4 continued*)

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{bmatrix} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\\ 1 & 1 & 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ y_1\\ z_1\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix} \quad (1.11)$$

namely $\{\lambda, \lambda_2, \lambda_3, \lambda_4\} = \{1, 0, 0, 0\}$ result.

*If* the point p is located arbitrarily, *then* the barycentric coordinates are represented by

$$\lambda_i = \frac{\Delta_i}{\Delta} \quad \forall i \in \{1, 2, 3, 4\} \quad (1.12)$$

where

$$\Delta_1 := \begin{vmatrix} X & X_2 & X_3 & X_4\\ Y & Y_2 & Y_3 & Y_4\\ Z & Z_2 & Z_3 & Z_4\\ 1 & 1 & 1 & 1\end{vmatrix} = \pm\, 6vol\{p, p_2, p_3, p_4\} \quad (1.13)$$

$$\Delta_2 := \begin{vmatrix} X_1 & X & X_3 & X_4\\ Y_1 & Y & Y_3 & Y_4\\ Z_1 & Z & Z_3 & Z_4\\ 1 & 1 & 1 & 1\end{vmatrix} = \pm\, 6vol\{p_1, p, p_3, p_4\} \quad (1.14)$$

$$\Delta_3 := \begin{vmatrix} X_1 & X_2 & X & X_4\\ Y_1 & Y_2 & Y & Y_4\\ Z_1 & Z_2 & Z & Z_4\\ 1 & 1 & 1 & 1\end{vmatrix} = \pm\, 6vol\{p_1, p_2, p, p_4\} \quad (1.15)$$

$$\Delta_4 := \begin{vmatrix} X_1 & X_2 & X_3 & X\\ Y_1 & Y_2 & Y_3 & Y\\ Z_1 & Z_2 & Z_3 & Z\\ 1 & 1 & 1 & 1\end{vmatrix} = \pm\, 6vol\{p_1, p_2, p_3, p\} \quad (1.16)$$

$$\Delta := \begin{vmatrix} X_1 & X_2 & X_3 & X_4\\ Y_1 & Y_2 & Y_3 & Y_4\\ Z_1 & Z_2 & Z_3 & Z_4\\ 1 & 1 & 1 & 1\end{vmatrix} = \pm\, 6vol\{p_1, p_2, p_3, p_4\} \quad (1.17)$$

where $\Delta_1, \Delta_2, \cdots, \Delta$ are (six times) the volume of the hyperfaces, i.e. the volumes of the respective tetrahedra $\{p, p_2, p_3, p_4\}, \{p_1, p, p_3, p_4\}, \cdots, \{p_1, p_2, p_3, p_4\}$. As special case we can represent the *Möbius barycentric coordinates* for an affine basis in $\mathscr{R}^2$ or $\mathscr{R}$, respectively, where $\lambda_i$ amount to the *area of the faces* or the *length of a line segment*, respectively. It has to be mentioned that the barycentric coordinates of a point depend on the chosen affine basis. For a given

*Continued*

(*Box 1.4 continued*)

point p, *if* we choose different sets of four points constituting an affine basis, *then* different barycentric coordinates of p will be obtained. The *affine basis* constituted in $\mathscr{R}, \mathscr{R}^2, \mathscr{R}^3$, respectively, is set up with two different points, three points which are not collinear, and four points which are not coplanar.
If we treat the length between the base points, the area of a face constituted by three base points and the volume of the tetrahedron hyperface constituted by four base points as unit, *then* the barycentric coordinates are essentially the length of a line segment, area of a triangle, volume of a tetrahedron hyperface, respectively. It is for that reason that the *triple coordinates* of a point $p \in \mathscr{R}^2$, namely, the *Möbius barycentric coordinates*, are called *area coordinates*, too.

## 2. P4P in terms of observables

By means of (1.7) we expressed the unknown position of the point $p \sim (x, y, z)$ in terms of *Möbius barycentric coordinates* $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ subject to $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$, namely with respect to an affine basis $\{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_1\}$. The benefit of such a transformation $(x, y, z) \Longrightarrow (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is taken as soon as we succeed to relate the *Möbius barycentric coordinates* via (1.12) $\lambda_i = \Delta_i/\Delta$ for all $i \in \{1, 2, 3, 4\}$ to the observables of type space angles $\psi_{i,j}$, the purpose of this paragraph.

From *analytical geometry* of $\mathbf{E}^3 := \{\mathscr{R}^3, g_{\mu\nu}\}$ we are used to represent, according to *Box 2.1*, a volume element $vol\{p, p_2, p_3, p_4\}$ by (2.1) as the projection of the *vector product* $\overrightarrow{pp_3} * \overrightarrow{pp_4}$ onto $\overrightarrow{pp_2}$. By taking advantage of the *Lagrange identity* we relate the *Cramer determinant* $\Delta_1^2 = (6\, vol\{p, p_2, p_3, p_4\})^2$ by (2.2) to the *inner products* $< \overrightarrow{pp_2} | \overrightarrow{pp_3} > = \|\overrightarrow{pp_2}\| \; \|\overrightarrow{pp_3}\| \cos\psi_{23}$, $< \overrightarrow{pp_2} | \overrightarrow{pp_2} > = \|\overrightarrow{pp_2}\|^2$ etc. Those inner products implemented into $\Delta_1^2, \cdots, \Delta_4^2$ via (2.3) lead to the representation of *Möbius barycentric coordinates* (2.4) in terms of the *observed space angles*, via (2.5), (2.6) finally written in the form (2.7).

Substituting $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}(\psi_{ij})$ of type (2.7) into (1.7) we finally arrive at the representations (2.8), (2.9), (2.10) $\{x, y, z\}(\psi_{ij})$, which solve P4P in $\mathbf{E}^3 := \{\mathscr{R}^3, g_{\mu\nu}\}$ in *Box 2.2.* (2.8), (2.9), (2.10) solving P4P in $\mathbf{E}^3 := \{\mathscr{R}^3, g_{\mu\nu}\}$ are generalization of the *Ansermet equations* (*A. Ansermet, 1910*) of *Appendix A*, which solve P3P in $\mathbf{E}^2 := \{\mathscr{R}^2, g_{\mu\nu}\}$. *But in contrast* to the *Ansermet equations* which are completely determined by the angular measurements $\psi_{ij}$, the equations (2.8), (2.9), (2.10) solving P4P in $\mathbf{E}^3$ depend also on the distances $\|\overrightarrow{pp_i}\|$ for all $i \in \{1, 2, 3, 4\}$. These distances $\|\overrightarrow{pp_i}\|$ are determined by an algorithm outlined in the *next paragraph*.

*Box 2.1:* The transformation of P4P observables of type space angles into *Möbius barycentric coordinates*

$$\Delta_1 = < \overrightarrow{pp_2} | \overrightarrow{pp_3} * \overrightarrow{pp_4} > = < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) * (\mathbf{x}_4 - \mathbf{x}) > \tag{2.1i}$$
$$\Delta_2 = < \overrightarrow{pp_1} | \overrightarrow{pp_3} * \overrightarrow{pp_4} > = < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) * (\mathbf{x}_4 - \mathbf{x}) > \tag{2.1ii}$$
$$\Delta_3 = < \overrightarrow{pp_1} | \overrightarrow{pp_2} * \overrightarrow{pp_4} > = < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) * (\mathbf{x}_4 - \mathbf{x}) > \tag{2.1iii}$$
$$\Delta_4 = < \overrightarrow{pp_1} | \overrightarrow{pp_2} * \overrightarrow{pp_3} > = < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) * (\mathbf{x}_3 - \mathbf{x}) > \tag{2.1iv}$$

"*Lagrange identity* (*Cramer determinant*)"

$$\Delta_1^2 = \begin{vmatrix} < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_4 - \mathrm{x}) > \\ < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \\ < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \end{vmatrix} \tag{2.2i}$$

$$\Delta_2^2 = \begin{vmatrix} < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \\ < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \\ < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \end{vmatrix} \tag{2.2ii}$$

$$\Delta_3^2 = \begin{vmatrix} < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \\ < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \\ < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_4 - \mathbf{x}) | (\mathbf{x}_4 - \mathbf{x}) > \end{vmatrix} \tag{2.2iii}$$

$$\Delta_4^2 = \begin{vmatrix} < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_1 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > \\ < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_2 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > \\ < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_1 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_2 - \mathbf{x}) > & < (\mathbf{x}_3 - \mathbf{x}) | (\mathbf{x}_3 - \mathbf{x}) > \end{vmatrix} \tag{2.2iv}$$

$$\Delta_1^2 = \|\mathbf{x}_2 - \mathbf{x}\|^2 \|\mathbf{x}_3 - \mathbf{x}\|^2 \|\mathbf{x}_4 - \mathbf{x}\|^2 \times (1 + 2\cos\psi_{23}\cos\psi_{34}\cos\psi_{42} - \cos^2\psi_{23} - \cos^2\psi_{34} - \cos^2\psi_{42}) \tag{2.3i}$$
$$\Delta_2^2 = \|\mathbf{x}_1 - \mathbf{x}\|^2 \|\mathbf{x}_3 - \mathbf{x}\|^2 \|\mathbf{x}_4 - \mathbf{x}\|^2 \times (1 + 2\cos\psi_{34}\cos\psi_{41}\cos\psi_{13} - \cos^2\psi_{34} - \cos^2\psi_{41} - \cos^2\psi_{13}) \tag{2.3ii}$$
$$\Delta_3^2 = \|\mathbf{x}_1 - \mathbf{x}\|^2 \|\mathbf{x}_2 - \mathbf{x}\|^2 \|\mathbf{x}_4 - \mathbf{x}\|^2 \times (1 + 2\cos\psi_{41}\cos\psi_{12}\cos\psi_{24} - \cos^2\psi_{41} - \cos^2\psi_{12} - \cos^2\psi_{24}) \tag{2.3iii}$$
$$\Delta_4^2 = \|\mathbf{x}_1 - \mathbf{x}\|^2 \|\mathbf{x}_2 - \mathbf{x}\|^2 \|\mathbf{x}_3 - \mathbf{x}\|^2 \times (1 + 2\cos\psi_{12}\cos\psi_{23}\cos\psi_{31} - \cos^2\psi_{12} - \cos^2\psi_{23} - \cos^2\psi_{31}) \tag{2.3iv}$$

$$\lambda_1 = \Delta_1/\Delta = (234)\|\mathbf{x}_2 - \mathbf{x}\| \ \|\mathbf{x}_3 - \mathbf{x} \ \|\mathbf{x}_4 - \mathbf{x}\|/\Delta \tag{2.4i}$$
$$\lambda_2 = \Delta_2/\Delta = (341)\|\mathbf{x}_3 - \mathbf{x}\| \ \|\mathbf{x}_4 - \mathbf{x} \ \|\mathbf{x}_1 - \mathbf{x}\|/\Delta \tag{2.4ii}$$
$$\lambda_3 = \Delta_3/\Delta = (412)\|\mathbf{x}_4 - \mathbf{x}\| \ \|\mathbf{x}_1 - \mathbf{x} \ \|\mathbf{x}_2 - \mathbf{x}\|/\Delta \tag{2.4iii}$$
$$\lambda_4 = \Delta_4/\Delta = (123)\|\mathbf{x}_1 - \mathbf{x}\| \ \|\mathbf{x}_2 - \mathbf{x} \ \|\mathbf{x}_3 - \mathbf{x}\|/\Delta \tag{2.4iv}$$

subject to

$$(234) := \pm\sqrt{1 + 2\cos\psi_{23}\cos\psi_{34}\cos\psi_{42} - \cos^2\psi_{23} - \cos^2\psi_{34} - \cos^2\psi_{42}} \tag{2.5i}$$
$$(341) := \pm\sqrt{1 + 2\cos\psi_{34}\cos\psi_{41}\cos\psi_{13} - \cos^2\psi_{34} - \cos^2\psi_{41} - \cos^2\psi_{13}} \tag{2.5ii}$$
$$(412) := \pm\sqrt{1 + 2\cos\psi_{41}\cos\psi_{12}\cos\psi_{24} - \cos^2\psi_{41} - \cos^2\psi_{12} - \cos^2\psi_{24}} \tag{2.5iii}$$
$$(123) := \pm\sqrt{1 + 2\cos\psi_{12}\cos\psi_{23}\cos\psi_{31} - \cos^2\psi_{12} - \cos^2\psi_{23} - \cos^2\psi_{31}} \tag{2.5iv}$$

(*Box 2.1 Continued*)

*if*

$$\delta := ||\mathbf{x}_1 - \mathbf{x}||\,||\mathbf{x}_2 - \mathbf{x}||\,||\mathbf{x}_3 - \mathbf{x}\,||\mathbf{x}_4 - \mathbf{x}||/\Delta \tag{2.6}$$

*then*

$$\lambda_1 = (234)\ \delta/||\mathbf{x}_1 - \mathbf{x}|| \quad \lambda_2 = (341)\ \delta/||\mathbf{x}_2 - \mathbf{x}|| \quad \lambda_3 = (412)\ \delta/||\mathbf{x}_3 - \mathbf{x}|| \quad \lambda_4 = (123)\ \delta/||\mathbf{x}_4 - \mathbf{x}|| \tag{2.7}$$

*Box 2.2:* The transformation of P4P observables of type space angles into Cartesian coordinates of the unknown point p, three dimensional analogous to the two-dimensional *Ansermet* (1910) algorithm of P3P in $\mathbf{E}^2 := \{\mathscr{R}^2, g_{\mu,v}\}$

$$\begin{aligned}
x &= \frac{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \\
&= \frac{(234)x_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)x_2/||\mathbf{x}_2 - \mathbf{x}|| + (412)x_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)x_4/||\mathbf{x}_4 - \mathbf{x}||}{(234)/||\mathbf{x}_1 - \mathbf{x}|| + (341)/||\mathbf{x}_2 - \mathbf{x}|| + (412)/||\mathbf{x}_3 - \mathbf{x}|| + (123)/||\mathbf{x}_4 - \mathbf{x}||} \\
&= \frac{||\mathbf{x}_1 - \mathbf{x}||\,||\mathbf{x}_2 - \mathbf{x}||\,||\mathbf{x}_3 - \mathbf{x}||\,||\mathbf{x}_4 - \mathbf{x}||}{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \end{vmatrix}} \\
&\quad \times [(234)x_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)x_2/||\mathbf{x}_2 - \mathbf{x}|| + (412)x_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)x_4/||\mathbf{x}_4 - \mathbf{x}||]
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
y &= \frac{\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4 \\
&= \frac{(234)y_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)y_2/||\mathbf{x}_2 - \mathbf{x}|| + (412)y_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)y_4/||\mathbf{x}_4 - \mathbf{x}||}{(234)/||\mathbf{x}_1 - \mathbf{x}|| + (341)/||\mathbf{x}_2 - \mathbf{x}|| + (412)/||\mathbf{x}_3 - \mathbf{x}|| + (123)/||\mathbf{x}_4 - \mathbf{x}||} \\
&= \frac{||\mathbf{x}_1 - \mathbf{x}||\,||\mathbf{x}_2 - \mathbf{x}||\,||\mathbf{x}_3 - \mathbf{x}||\,||\mathbf{x}_4 - \mathbf{x}||}{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \end{vmatrix}} \\
&\quad \times [(234)y_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)y_2/||\mathbf{x}_2 - \mathbf{x}|| + (412)y_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)y_4/||\mathbf{x}_4 - \mathbf{x}||]
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
z &= \frac{\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 \\
&= \frac{(234)z_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)z_2/||\mathbf{x}_2 - \mathbf{x}|| + (412)z_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)z_4/||\mathbf{x}_4 - \mathbf{x}||}{(234)/||\mathbf{x}_1 - \mathbf{x}|| + (341)/||\mathbf{x}_2 - \mathrm{x}|| + (412)/||\mathbf{x}_3 - \mathbf{x}|| + (123)/||\mathbf{x}_4 - \mathbf{x}||} \\
&= \frac{||\mathbf{x}_1 - \mathbf{x}||\ ||\mathbf{x}_2 - \mathbf{x}||\ ||\mathbf{x}_3 - \mathbf{x}||\ ||\mathbf{x}_4 - \mathbf{x}||}{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \end{vmatrix}} \\
&\quad \times [(234)z_1/||\mathbf{x}_1 - \mathbf{x}|| + (341)z_2/||\mathbf{x}_2 - \mathbf{x}\ || + (412)z_3/||\mathbf{x}_3 - \mathbf{x}|| + (123)z_4/||\mathbf{x}_4 - \mathbf{x}||]
\end{aligned} \tag{2.10}$$

## 3. P4P in terms of distances and observables of type space angles

Within the *five-dimensional simplex* $\{p, p_1, p_2, p_3, p_4\}$ the space angle $\psi_{ij}$ of the faces $\{p_i p p_j\}, i \neq j$, centred at p are related by (3.1) to the distances $||\overrightarrow{pp_i}||$, $||\overrightarrow{pp_j}||$, $||\overrightarrow{p_ip_j}||$ by means of the *side cosine theorem,* namely constructing the *Grunert equations* (Grunert, 1841) (3.3), with respect to the coefficients $a_{ij}, b_{ij}$, respectively, defined by means of (3.2) proportional to the cosine of space angles $\psi_{ij}$, the distances $||\overrightarrow{p_ip_j}||$, respectively. One set of *Grunert equations,* e.g. (3.3ii), (3.3iii), (3.3vi), is solved. The geometric background of these quadratic forms in $\{x_2, x_3\}$, $\{x_3, x_4\}, \{x_2, x_4\}$, one dimensional manifolds imbeded in $\mathscr{R}^+ \times \mathscr{R}^+ \times \mathscr{R}^+$, in general, is described in *E. Grafarend, et al* (1989) in detail. Here we map a point in $\{\mathscr{R}^{+3}, g_{\mu,\nu}\}$ by a projection line onto the *Grassmann manifold* $G_{3,2}(x_2 = 1)$ introducing the homogeneous coordinates $(3.4iii) y_3 := x_3/x_2, y_4 := x_4/x_2$, respectively, as outlined in *Example 1.1 and 1.2.*

The *forward algorithm* in solving three *quadratic equations* in the unknowns $x_i \in \mathscr{R}^+$ for all $i \in \{2, 3, 4\}$, namely the *Grunert equations* of Box. 3.1, starts with the transformation of three *inhomogeneous quadratic forms* into three *homogeneous quadratic forms* by projective geometry as the *first step.* The second and third steps are based upon the reduction of the homogeneous set of equations towards *one equation of four order,* namely by expressing $y_4(y_3)$ (3.7). In the *fourth step* (3.7) is replaced in (3.6ii) leading to the *four order equations* (3.8), (3.9) for $y_4$ which may at most have *four positive roots.*

In the *backward algorithm* we use the at most *four positive roots* of the four order equation (3.8) of $y_3$ to successively compute (*1st step*) $x_2$ via $y_3$, (3.5), (*2nd step*) $x_3$ via $y_3, x_2$ (3.4iv), (*3rd step*) $x_4$ via $y_3, y_4$, (3.7) and (3.4iv) and finally (*4th step*) *test the validity* of the at most four positive roots $\{x_2, x_3, x_4\}_{1,2,3,4}$ with respect to consistency of the equations (3.3i), (3.3iv) and (3.3v).

To ensure high computational stability in the calculation of the at most four positive roots of the *Grunert equations,* those three *Grunert* equations or three points of known positions should be chosen which have the *best configuration*: Three points of known position are to be selected which guarantee *maximum summation of the face angles* $\psi_{ij}$ with respect to the unknown point.

*Box 3.1:* The *Grunert* equations (*Grunert,* 1841) relating known space angles $\psi_{ij}$ to unknown distances $||\overrightarrow{pp_i}||, ||\overrightarrow{pp_j}||$ and known distances $||\overrightarrow{p_ip_j}||$ for all $i, j \in \{1, 2, 3, 4\}, i \neq j$.

$$||\overrightarrow{p_1p_2}||^2 = ||\overrightarrow{pp_1}||^2 + ||\overrightarrow{pp_2}||^2 - 2||\overrightarrow{pp_1}||\ ||\overrightarrow{pp_2}|| \cos\psi_{12} \tag{3.1i}$$

$$||\overrightarrow{p_2p_3}||^2 = ||\overrightarrow{pp_2}||^2 + ||\overrightarrow{pp_3}||^2 - 2||\overrightarrow{pp_2}||\ ||\overrightarrow{pp_3}|| \cos\psi_{23} \tag{3.1ii}$$

$$||\overrightarrow{p_3p_4}||^2 = ||\overrightarrow{pp_3}||^2 + ||\overrightarrow{pp_4}||^2 - 2||\overrightarrow{pp_3}||\ ||\overrightarrow{pp_4}|| \cos\psi_{34} \tag{3.1iii}$$

$$||\overrightarrow{p_4p_1}||^2 = ||\overrightarrow{pp_4}||^2 + ||\overrightarrow{pp_1}||^2 - 2||\overrightarrow{pp_4}||\ ||\overrightarrow{pp_1}|| \cos\psi_{41} \tag{3.1iv}$$

$$||\overrightarrow{p_1p_3}||^2 = ||\overrightarrow{pp_1}||^2 + ||\overrightarrow{pp_3}||^2 - 2||\overrightarrow{pp_1}||\ ||\overrightarrow{pp_3}|| \cos\psi_{13} \tag{3.1v}$$

$$||\overrightarrow{p_2p_4}||^2 = ||\overrightarrow{pp_2}||^2 + ||\overrightarrow{pp_4}||^2 - 2||\overrightarrow{pp_2}||\ ||\overrightarrow{pp_4}|| \cos\psi_{24} \tag{3.1vi}$$

$$||\overrightarrow{p_1p_2}||^2 =: b_{12} \quad ||\overrightarrow{p_2p_3}||^2 =: b_{23} \quad ||\overrightarrow{p_3p_4}||^2 =: b_{34}$$

$$||\overrightarrow{p_4p_1}||^2 =: b_{41} \quad ||\overrightarrow{p_1p_3}||^2 =: b_{13} \quad ||\overrightarrow{p_2p_4}||^2 =: b_{24}$$

$$a_{12} := -2\cos\psi_{12} \quad a_{23} := -2\cos\psi_{23} \quad a_{34} := -2\cos\psi_{34}$$

$$a_{41} := -2\cos\psi_{41} \quad a_{13} := -2\cos\psi_{13} \quad a_{24} := -2\cos\psi_{24}$$

$$||\overrightarrow{pp_1}|| := x_1 \quad ||\overrightarrow{pp_2}|| := x_2 \quad ||\overrightarrow{pp_3}|| := x_3 \quad ||\overrightarrow{pp_4}|| := x_4 \tag{3.2}$$

*Continued*

(*Box 3.1 Continued*)

$$\begin{aligned} x_1^2 + x_2^2 + a_{12}x_1x_2 &= b_{12} \\ x_2^2 + x_3^2 + a_{23}x_2x_3 &= b_{23} \quad & -2 \le a_{ij} \le +2 \\ x_3^2 + x_4^2 + a_{34}x_3x_4 &= b_{34} \quad & x_i \in \mathscr{R}^+ \\ x_4^2 + x_1^2 + a_{41}x_4x_1 &= b_{41} \quad & b_{ij} \in \mathscr{R}^+ \\ x_1^2 + x_3^2 + a_{13}x_1x_3 &= b_{13} \quad & \forall i,j \in \{1,2,3,4\} \\ x_2^2 + x_4^2 + a_{24}x_2x_4 &= b_{24} \end{aligned} \tag{3.3}$$

*"One set of Grunert equation* (*3.3ii*) (*3.3iii*), (*3.3vi*)"

$$x_2^2 + x_3^2 + a_{23}x_2x_3 = b_{23}$$
$$x_3^2 + x_4^2 + a_{34}x_3x_4 = b_{34}$$
$$x_2^2 + x_4^2 + a_{24}x_2x_4 = b_{24}$$

*The Grunert algorithm*

*The forward algorithm*

1st step

$$x_2^2\left(1 + \frac{x_3^2}{x_2^2} + a_{23}\frac{x_3}{x_2}\right) = b_{23} \tag{3.4i}$$

$$x_2^2\left(\frac{x_3^2}{x_2^2} + \frac{x_4^2}{x_2^2} + a_{34}\frac{x_3x_4}{x_2x_2}\right) = b_{34} \tag{3.4ii}$$

$$x_2^2\left(1 + \frac{x_4^2}{x_2^2} + a_{24}\frac{x_4}{x_2}\right) = b_{24} \tag{3.4iii}$$

$$\frac{x_3}{x_2} =: y_3 \quad \frac{x_4}{x_2} =: y_4 \tag{3.4iv}$$

(*homogeneous coordinates*)

2nd step

$$\begin{aligned} x_2^2 &= b_{23}/(1 + y_3^2 + a_{23}y_3) \\ &= b_{34}/(y_3^2 + y_4^2 + a_{34}y_3y_4) \\ &= b_{24}/(1 + y_4^2 + a_{24}y_4) \end{aligned} \tag{3.5}$$

(3.4ii)/(3.4i), (3.4iii)/(3.4i) :

$$b_{34}(1 + y_3^2 + a_{23}y_3) = b_{23}(y_3^2 + y_4^2 + a_{34}y_3y_4) \tag{3.6i}$$

$$b_{24}(1 + y_3^2 + a_{23}y_3) = b_{23}(1 + y_4^2 + a_{24}y_4) \tag{3.6ii}$$

3rd step

(3.6ii)-(3.6i)

*Continued*

(*Box 3.1 Continued*)

$$\begin{aligned} (b_{34} - b_{23})y_3^2 - b_{23}y_4^2 - a_{34}b_{23}y_3y_4 \\ + a_{23}b_{34}y_3 + b_{34} &= 0 \\ b_{24}y_3^2 - b_{23}y_4^2 + a_{23}b_{24}y_3 \\ - a_{24}b_{23}y_4 + b_{24} - b_{23} &= 0 \\ (b_{34} - b_{23} - b_{24})y_3^2 - a_{34}b_{23}y_3y_4 \\ + a_{23}(b_{34} - b_{24})y_3 + a_{24}b_{23}y_4 \\ + b_{34} - b_{24} + b_{23} &= 0 \end{aligned}$$

$$y_4 = \frac{(b_{34} - b_{23} - b_{24})y_3^2 + a_{23}(b_{34} - b_{24})y_3}{a_{34}b_{23}y_3 - a_{24}b_{23}} + \frac{(b_{23} + b_{34} - b_{24})}{a_{34}b_{23}y_3 - a_{24}b_{23}} \tag{3.7}$$

4th step

(3.7)$\Longrightarrow$(3.6ii)

$$C_4y_3^4 + C_3y_3^3 + C_2y_3^2 + C_1y_3 + C_0 = 0 \tag{3.8}$$

subject to

$$C_4 := b_{23}(b_{34} - b_{23} - b_{24})^2 - a_{34}^2b_{23}^2b_{24} \tag{3.9i}$$

$$\begin{aligned} C_3 := (b_{34} - b_{23} - b_{24}) \\ \times\ [a_{24}a_{34}b_{23}^2 + 2a_{23}b_{23}(b_{34} - b_{24})] \\ +\ a_{34}b_{23}^2b_{24}(2a_{24} - a_{23}a_{34}) \end{aligned} \tag{3.9ii}$$

$$\begin{aligned} C_2 := b_{23}[a_{23}^2(b_{34} - b_{24})^2 \\ + 2(b_{34} - b_{23} - b_{24})\ (b_{23} + b_{34} - b_{24})] \\ +\ a_{23}a_{24}a_{34}b_{23}^2(b_{24} + b_{34}) + a_{34}^2b_{23}^2(b_{23} - b_{24}) \\ a_{24}^2b_{23}^2(b_{23} - b_{34}) \end{aligned} \tag{3.9iii}$$

$$\begin{aligned} C_1 := 2a_{23}b_{23}(b_{34} - b_{24})(b_{23} + b_{34} - b_{24}) \\ +\ a_{24}a_{34}b_{23}^2(b_{34} + b_{24} - b_{23}) \\ -\ a_{23}a_{24}^2b_{23}^2b_{34} \end{aligned} \tag{3.9iv}$$

$$C_0 := b_{23}(b_{23} + b_{34} - b_{24})^2 - a_{24}^2b_{23}^2b_{34} \tag{3.9v}$$

The backward algorithm

1st step

insert $y_3$ into (3.5) in order to obtain $x_2$

2nd step

insert $y_3$ and $x_2$ into (3.4iv) in order to obtain $x_3$

3rd step

insert $y_3$ into (3.7) to obtain $y_4$, insert $y_4$ into (3.4iv) to obtain $x_4$

*Continued*

(*Box 3.1 Continued*)

4th step

use all at most four positive solutions of (3.8) to compute $\{x_2, x_3, x_4\}_{1,2,3,4}$ and insert the at most four positive solutions into (3.3*i*), (3.3*iv*) *and* (3.3*v*) *to test validity*.

## 4. P4P, an example

A practical verification of the P4P is the following. *Table 1.* is a list of $\{x, y, z\}$ coordinates of four given points in $\{\mathscr{R}^3, g_{\mu,\nu}\}$. In contrast, *Table 2.* summarizes the observations of type space angles, namely $\cos\psi_{ij}$. Following the *P4P algorithm* described in the pre ceding chapters, we have computed the intermediate quantities $||\overrightarrow{pp_1}|| = ||\mathbf{x}_1 - \mathbf{x}||$, $||\overrightarrow{pp_2}|| = ||\mathbf{x}_2 - \mathbf{x}||$, $||\overrightarrow{pp_3}|| = ||\mathbf{x}_3 - \mathbf{x}||, ||\overrightarrow{pp_4}|| = ||\mathbf{x}_4 - \mathbf{x}||$ as distances from the *Grunert equations,* the substitutional quantities $\delta, (234), (341), (412), (123)$ and the *Möbius barycentric coordinates* are listed in *Table 3.* Finally *Table 4.* lists the $\{x, y, z\}$ coordinates of the unknown point as being computed from (2.8), (2.9), (2.10).

The P4P algorithm

Step 1. Compute the *space angles* $\psi_{ij}$ of the five-dimensional simplex $\{p, p_1, p_2, p_3, p_4\}$ from *either* (1.1) *or* (1.2) relating space angles to original observations of type "image coordinates" *or "horizontal and vertical direction"*.
Step 2. Compute the *distances* $||\mathbf{x}_i - \mathbf{x}_j||$ from given Cartesian coordinates of the point $p_i, p_j$, respectively.
Step 3. Solve the *Grunert* equations (3.3) by the algorithm (3.4)–(3.9).
Step 4. Compute the Cartesian coordinates of the unknown point $p$ by means of (2.8)–(2.10) subject to the bracket terms (2.5): (234), (341), (412), (123).

**Tab 1**. P4P, Cartesian coordinates of four given points

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ |
|---|---|---|---|---|
| x | 0 | 2 | 2 | 0 |
| y | 0 | 0 | 2 | 2 |
| z | 1 | –1 | 1 | –1 |

**Tab 2**. P4P, observables of type space angles

| | | | |
|---|---|---|---|
| $\cos\psi_{12}$ | 0.870388 | $\cos\psi_{23}$ | 0.870388 |
| $\cos\psi_{34}$ | 0.870388 | $\cos\psi_{41}$ | 0.870388 |
| $\cos\psi_{13}$ | 0.636364 | $\cos\psi_{24}$ | 0.636364 |

**Tab 3**. P4P, intermediate quantities, solutions of the *Grunert* equations

| | |
|---|---|
| $\|\|x_1 - x\|\| = 3.3166$ | $\|\|x_2 - x\|\| = 5.1962$ |
| $\|\|x_3 - x\|\| = 3.3166$ | $\|\|x_4 - x\|\| = 5.1962$ |
| $\delta = 18.5625$ | |
| (234) = (412) = ± 0.2233 | (341) = (123) = ±0.2099 |
| $\lambda_1 = 1.25$ $\lambda_2 = -0.75$ | $\lambda_3 = 1.25$ $\lambda_4 = -0.75$ |

**Tab 4**. P4P, Cartesian coordinates of the unknown point

| | | |
|---|---|---|
| x = 1.0000 | y = 1.0000 | z = 4.0000 |

## Appendix A.

*The Ansermet algorithm solving P3P in* $\mathbf{E}^2 = \{\mathscr{R}^2, g_{\mu\nu}\}$

As soon as we reduce the perspective 4-point problem in the three-dimensional Euclidean manifold $\mathbf{E}^3 = \{\mathscr{R}^3, g_{\mu,\nu}\}$ to the perspective 3-point problem in the two-dimensional Euclidean manifold $\mathbf{E}^2 = \{\mathscr{R}^2, g_{\mu\nu}\}$ (in accordance with the *PnP* notation we shortly denote this problem here as *P3P problem in two dimentions*), we are led to the *Möbius barycentric coordinates* (A1) of *Box A1* subject to (A2) or (A3) illustrated by *Figure A1.* Correspondingly by specifying (1.7) we transform the *Möbius barycentric coordinates* $\{\lambda_1, \lambda_2, \lambda_3\}$ into the Cartesian coordinates $\{x, y\}$ by means of (A4) and (A5) of *Box A2*. Implementing $\Delta = 2\ area\{p_1, p_2, p_3\}$ (A3) into (A4), (A5) leads to (A6), (A7) similar to (2.8)–(2.10) as a representation of Cartesian coordinates $\{x, y\}$ in terms of the observables of the type angles $\{\psi_{23}, \psi_{31}, \psi_{12}\}$ and given Cartesian coordinates $\{x_1, y_1\} \sim p_1, \{x_2, y_2\} \sim p_2, \{x_3, y_3\} \sim p_3$. Indeed (A6), (A7) can only be made operational as soon as we are informed of $\|\mathbf{x}_1 - \mathbf{x}\|, \|\mathbf{x}_2 - \mathbf{x}\|, \|\mathbf{x}_3 - \mathbf{x}\|$, for instance by solving the corresponding *Grunert equations* (3.3) subject to $\psi_{12} + \psi_{23} + \psi_{31} = 2\pi$.

An alternative solution of P3P in two dimensions is based upon the *C.F. Gauß criterion* (*C.F. Gauß,* 1842, *A. Schreiber,* 1908), namely the computation of Ω defined by (A8) which is the mutual *product sum of* $<\overrightarrow{p_1p_2}|\overrightarrow{p_1p_3}>$ *area*$\{p, p_2, p_3\}$ and $<\overrightarrow{pp_2}|\overrightarrow{pp_3}>$ *area* $\{p_1, p_2, p_3\}$. Actually the *product sum* generates a constant which does not change for a cyclic permutation of indices. For instance, an alternative representation of the *product sum* Ω is $<\overrightarrow{p_2p_3}|\overrightarrow{p_2p_1}>$ *area*$\{p, p_1, p_3\}$ and $<\overrightarrow{pp_3}|\overrightarrow{pp_1}>$ *area*$\{p_1, p_2, p_3\}$ or $<\overrightarrow{p_3p_1}|\overrightarrow{p_3p_2}>$ *area* $\{p, p_1, p_2\}$ and $<\overrightarrow{pp_1}|\overrightarrow{pp_3}>$ *area*$\{p_1, p_2, p_3\}$. As soon as we represent the elements of area by $2\ area\ \{p, p_2, p_3\} = \|\mathbf{x}_2 - \mathbf{x}\|\|\mathbf{x}_3 - \mathbf{x}\|\ \sin\psi_{23}$ or $2\ area\ \{p_1, p_2, p_3\} = \|\mathbf{x}_2 - \mathbf{x}_1\|\|\mathbf{x}_3 - \mathbf{x}_1\|\ \sin\psi_{213}$ by means of (A9)–(A11), we generate the *Möbius barycentric coordinates* (A13) as being proportional to $\lambda_1 \sim (\cot\psi_{213} - \cot\psi_{23})^{-1}$, $\lambda_2 \sim (\cot\psi_{321} - \cot\psi_{31})^{-1}$,

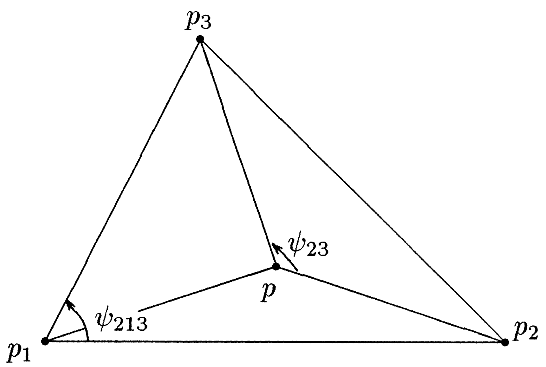


**Fig. 1A.** P3P in $\mathbf{E}^2 := \{\mathscr{R}^2, g_{\mu\nu}\}$, *Ansermet* barycentric coordinates $\angle p_2p_1p_3 = \psi_{213}$, $\angle p_2pp_3 = \psi_{23}$

$\lambda_3 \sim (\cot\psi_{132} - \cot\psi_{12})^{-1}$ respectively. Finally we are led to the *Cartesian coordinates* {x,y} of (A14), (A15) as the *weighted mean* of the given *Cartesian coordinates* $\{x_1, y_1\} \sim p_1$, $\{x_2, y_2\} \sim p_2$, $\{x_3, y_3\} \sim p_3$, the weights defined by (A16) subject to (A17). (A14)–(A16) are the *Ansermet barycentric coordinates* $g_1, g_2, g_3$ transformed into the *Cartesian coordinates* $\{x, y\}$ of the *unknown* point $\{x, y\} \sim p$.

*Box A1:* The transformation of P3P observables of type angles into *Möbius barycentric coordinates.*

$$\lambda_1 = \frac{\Delta_1}{\Delta} = \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \sin\psi_{23}/\Delta \quad \text{(A1i)}$$

$$\lambda_2 = \frac{\Delta_2}{\Delta} = \|\mathbf{x}_3 - \mathbf{x}\| \, \|\mathbf{x}_1 - \mathbf{x}\| \sin\psi_{31}/\Delta \quad \text{(A1ii)}$$

$$\lambda_3 = \frac{\Delta_3}{\Delta} = \|\mathbf{x}_1 - \mathbf{x}\| \, \|\mathbf{x}_2 - \mathbf{x}\| \sin\psi_{12}/\Delta \quad \text{(A1iii)}$$

subject to

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \quad \text{(A2)}$$

or

$$\Delta = \|\mathbf{x}_1 - \mathbf{x}\| \, \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \times \left[\frac{\sin\psi_{23}}{\|\mathbf{x}_1 - \mathbf{x}\|} + \frac{\sin\psi_{31}}{\|\mathbf{x}_2 - \mathbf{x}\|} + \frac{\sin\psi_{12}}{\|\mathbf{x}_3 - \mathbf{x}\|}\right] \quad \text{(A3)}$$

## Appendix B. Solution for orientation elements

Given the Cartesian coordinates of those points which build up the *five dimensional simplex* $\{p, p_1, p_2, p_3, p_4\}$, we are left with the problem to determine the *orientation elements* of the observational reference frame of the theodolite, the camera or the CCD sensor. These *orientation elements* will be needed for solving the *three dimensional intersection problem* from two perspective centres which are localized by the solution vectors of the *twin* P4P problem, for instance. The starting point is the representation of the relative placement vector $\mathbf{x}_i - \mathbf{x}$ in *spherical coordinates* $\{\alpha_i, \beta_i, \|\overrightarrow{pp_i}\|\}$ in case of {horizontal directions, vertical directions, distances} taken in a *local level theodolite frame* and a subsequent rotation from the *local level theodolite frame* into the *global reference frame* in which the Cartesian coordinates of $p, p_i$ for all $i \in \{1, 2, 3, 4\}$ have been given or computed. Alternatively the relative placement vector $\mathbf{x}_i - \mathbf{x}$ is represented in Cartesian coordinates $\{\bar{x}_i, \bar{y}_i, -f\}$ taken in a *local camera/CCD sensor frame* relative to the perspective centre and a subsequent rotation from the *local camera/CCD sensor frame* into the *global reference frame* described above. An outline of the fundamental coordinate transformations is given in *Box B2* in terms of image coordinates of a *local camera/CCD sensor frame,* and in *Box B1* in terms of spherical coordinates of a *local level theodolite frame.*

(B1) and (B11) represent the basic transformations which are inconsistent equations due to measurement inconsistenties within the observations $\{\alpha_i, \beta_i\}$ and $\{\bar{x}_i, \bar{y}_i\}$ respectively. The orthogonal matrix $\mathbf{R} \in \mathbf{SO}(3)$ has been decomposed according to the *Lipschitz representation* ("unit quaternions") as the *Cayley product* of the matrices $(\mathbf{I} - \mathbf{S})^{-1}$ and $(\mathbf{I} + \mathbf{S})$ where $\mathbf{S}$ is a *skew matrix*: (B2), (B3). The transpose matrix $\mathbf{S}^T$ has been vectorized by means of (B3iii) $\mathrm{vec}\mathbf{S}^T \in \mathscr{R}^{9\times 1}$ which in turn by means of the matrix $\mathbf{K} \in \mathbf{Z}^{9\times 3}$ has been mapped onto $\mathrm{vec}\,k\mathbf{S}^T = [a\ b\ c]^T$ (*read*: "vecskew"), namely, the essential elements of the *antisymmetric matrix* $\mathbf{S}^T$. In order to determine the three orientation elements (a, b, c) one set of equations would be sufficient, for instance those generated by $(x_1 - x, y_2 - y, z_3 - z)$. Instead we are proposing to use all the 4 points within the P4P to calculate the three orientation elements (a, b, c) by means of adjustment (B8), (B9), so as to generate a more reliable solution. For this purpose (B4), (B12) contain, as a matrix equation for $\mathbf{S}$, all point coordinates $(x_i - x, y_i - y, z_i - z)$ for all $i \in \{1, 2, 3, 4\}$. Those equations are constructed from (B1), (B11) by implementing (B2), (B3) and ordering a set of *inhomogeneous matrix equations.* A final vecterization based upon the *Kronecker product* "$\otimes$," namely, $\mathrm{vec}(\mathbf{AB}) = (\mathbf{I}_n \otimes \mathbf{A})\mathrm{vec}\mathbf{B} = (\mathbf{B}^T \otimes \mathbf{I}_l)\mathrm{vec}\mathbf{A}$ for all $\mathbf{A} \in \mathscr{R}^{l\times m}$, $\mathbf{B} \in \mathscr{R}^{m\times n}$ (see *E. Grafarend and B. Schaffrin* (1993, p. 419) for instance) followed by the *essential elements* mapping $\mathrm{vec}\mathbf{B} = \mathbf{K}\mathrm{veck}\mathbf{B}$, if $\mathbf{B} = -\mathbf{B}^T$, leads to the inconsistent equations (B8) solved into (B9) by "*least squares*".

*Box A2:* The transformation of P3P observables of type angles into Cartesian coordinates of the unknown point p, the *A. Ansermet* (1910) algorithm, P3P in $\mathbf{E}^2 = \{\mathscr{R}^2, g_{\mu\nu}\}$

$$x = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1 + \lambda_2 + \lambda_3} = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$
$$= \frac{\|\mathbf{x}_1 - \mathbf{x}\| \, \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\|}{\Delta} \left[ \frac{\sin\psi_{23}}{\|\mathbf{x}_1 - \mathbf{x}\|} x_1 + \frac{\sin\psi_{31}}{\|\mathbf{x}_2 - \mathbf{x}\|} x_2 + \frac{\sin\psi_{12}}{\|\mathbf{x}_3 - \mathbf{x}\|} x_3 \right] \tag{A4}$$

$$y = \frac{\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3}{\lambda_1 + \lambda_2 + \lambda_3} = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$$
$$= \frac{\|\mathbf{x}_1 - \mathbf{x}\| \, \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\|}{\Delta} \left[ \frac{\sin\psi_{23}}{\|\mathbf{x}_1 - \mathbf{x}\|} y_1 + \frac{\sin\psi_{31}}{\|\mathbf{x}_2 - \mathbf{x}\|} y_2 + \frac{\sin\psi_{12}}{\|\mathbf{x}_3 - \mathbf{x}\|} y_3 \right] \tag{A5}$$

$$x = \frac{\sin\psi_{23} x_1/\|\mathbf{x}_1 - \mathbf{x}\| + \sin\psi_{31} x_2/\|\mathbf{x}_2 - \mathbf{x}\| + \sin\psi_{12} x_3/\|\mathbf{x}_3 - \mathbf{x}\|}{\sin\psi_{23}/\|\mathbf{x}_1 - \mathbf{x}\| + \sin\psi_{31}/\|\mathbf{x}_2 - \mathbf{x}\| + \sin\psi_{12}/\|\mathbf{x}_3 - \mathbf{x}\|} \tag{A6}$$

$$y = \frac{\sin\psi_{23} y_1/\|\mathbf{x}_1 - \mathbf{x}\| + \sin\psi_{31} y_2/\|\mathbf{x}_2 - \mathbf{x}\| + \sin\psi_{12} y_3/\|\mathbf{x}_3 - \mathbf{x}\|}{\sin\psi_{23}/\|\mathbf{x}_1 - \mathbf{x}\| + \sin\psi_{31}/\|\mathbf{x}_2 - \mathbf{x}\| + \sin\psi_{12}/\|\mathbf{x}_3 - \mathbf{x}\|} \tag{A7}$$

“The Gauβ criterion”

$$\begin{aligned}
\Omega: &= < \overrightarrow{p_1p_2} | \overrightarrow{p_1p_3} > area\{p, p_2, p_3\} + < \overrightarrow{pp_2} | \overrightarrow{pp_3} > area\{p_1, p_2, p_3\} \\
&= (< \mathbf{x}_2 - \mathbf{x}_1 | \mathbf{x}_3 - \mathbf{x}_1 > \Delta_1 + < \mathbf{x}_2 - \mathbf{x} | \mathbf{x}_3 - \mathbf{x} > \Delta_2)/2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\| \, \|\mathbf{x}_3 - \mathbf{x}_1\| \, \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \left[ \cos\psi_{213} \sin\psi_{23} + \sin\psi_{213} \cos\psi_{23} \right] \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\| \, \|\mathbf{x}_3 - \mathbf{x}_1\| \, \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \sin(\psi_{213} - \psi_{23})
\end{aligned} \tag{A8}$$

$$\|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| = \frac{\Omega}{\|\mathbf{x}_2 - \mathbf{x}_1\| \, \|\mathbf{x}_3 - \mathbf{x}_1\| \sin(\psi_{213} - \psi_{23})} \tag{A9}$$

$$\Delta_1 = \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \sin\psi_{23} \tag{A10}$$
$$\Delta = \|\mathbf{x}_2 - \mathbf{x}_1\| \, \|\mathbf{x}_3 - \mathbf{x}_1\| \sin\psi_{213} \tag{A11}$$

$$\begin{aligned}
\lambda_1 &= \frac{\Delta_1}{\Delta} = \frac{area\{p, p_2, p_3\}}{area\{p_1, p_2, p_3\}} = \|\mathbf{x}_2 - \mathbf{x}\| \, \|\mathbf{x}_3 - \mathbf{x}\| \; \sin\psi_{23}/\Delta \\
&= \frac{\Omega}{\Delta} \frac{\sin\psi_{23}}{\sin(\psi_{213} - \psi_{23})} \frac{1}{\|\mathbf{x}_2 - \mathbf{x}_1\| \, \|\mathbf{x}_3 - \mathbf{x}_1\|} \\
&= \frac{\Omega}{\Delta^2} \frac{\sin\psi_{23} \sin\psi_{213}}{\sin(\psi_{213} - \psi_{23})} = \frac{\Omega}{\Delta^2} \frac{1}{\cot\psi_{213} - \cot\psi_{23}}
\end{aligned} \tag{A12}$$

$$\lambda_1 = \frac{\Omega}{\Delta^2} \frac{1}{\cot\psi_{213} - \cot\psi_{23}} \quad \lambda_2 = \frac{\Omega}{\Delta^2} \frac{1}{\cot\psi_{321} - \cot\psi_{31}} \quad \lambda_3 = \frac{\Omega}{\Delta^2} \frac{1}{\cot\psi_{132} - \cot\psi_{12}} \tag{A13}$$

$$x = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{g_1 x_1 + g_2 x_2 + g_3 x_3}{g_1 + g_2 + g_3} \tag{A14}$$

*Continued*

(*Box A.2 Continued*)

$$y = \frac{\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{g_1 y_1 + g_2 y_2 + g_3 y_3}{g_1 + g_2 + g_3} \tag{A15}$$

subject to

$$\begin{aligned} g_1 &:= \frac{1}{\cot\psi_{213} - \cot\psi_{23}} \\ g_2 &:= \frac{1}{\cot\psi_{321} - \cot\psi_{31}} \\ g_3 &:= \frac{1}{\cot\psi_{132} - \cot\psi_{12}} \end{aligned} \tag{A16}$$

$$g_1 + g_2 + g_3 = \frac{\Delta^2}{\Omega} \tag{A17}$$

*Box B.1:* Transformation between observed coordinates of the relative placement vector $\mathbf{x}_i - \mathbf{x}$ in a local frame and relative Cartesian coordinates in a global reference frame, *case* (*i*): spherical coordinates $\{\alpha_i, \beta_i\}$ of type horizontal and vertical direction in a *local level theodolite frame.*

$$\begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix} = \mathbf{R}\|\overrightarrow{pp_i}\| \begin{bmatrix} \cos\alpha_i \cos\beta_i \\ \sin\alpha_i \cos\beta_i \\ \sin\beta_i \end{bmatrix} = \mathbf{R}\|\overrightarrow{pp_i}\| \begin{bmatrix} ex_i \\ ey_i \\ ez_i \end{bmatrix} \quad \forall i \in \{1,2,3,4\} \tag{B1}$$

"for details consult *E. Grafarend, P. Lohse and B. Schaffrin* (1989, pp. 172–175)"

$$\mathbf{SO}(3) \ni \mathbf{R} = (\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S}) \tag{B2}$$

subject to

$$\mathbf{S} = -\mathbf{S}^T = \begin{bmatrix} 0 & +a & +b \\ -a & 0 & +c \\ -b & -c & 0 \end{bmatrix} \tag{B3i}$$

$$\text{veck}\mathbf{S}^T := [a\ b\ c]^T \tag{B3ii}$$

$$\text{vec}\mathbf{S}^T = \mathbf{K}\text{veck}\mathbf{S}^T \tag{B3iii}$$

subject to

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}^T \in \mathbf{Z}^{9\times 3} \tag{B3iv}$$

*Continued*

(*Box B.1 continued*)

$$\mathbf{Y} := \begin{bmatrix} x_1 - x - \|\overrightarrow{pp_1}\|e_{x_1} & y_1 - y - \|\overrightarrow{pp_1}\|e_{y_1} & z_1 - z - \|\overrightarrow{pp_1}\|e_{z_1} \\ x_2 - x - \|\overrightarrow{pp_2}\|e_{x_2} & y_2 - y - \|\overrightarrow{pp_2}\|e_{y_2} & z_2 - z - \|\overrightarrow{pp_2}\|e_{z_2} \\ x_3 - x - \|\overrightarrow{pp_3}\|e_{x_3} & y_3 - y - \|\overrightarrow{pp_3}\|e_{y_3} & z_3 - z - \|\overrightarrow{pp_3}\|e_{z_3} \\ x_4 - x - \|\overrightarrow{pp_4}\|e_{x_4} & y_4 - y - \|\overrightarrow{pp_4}\|e_{y_4} & z_4 - z - \|\overrightarrow{pp_4}\|e_{z_4} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x + \|\overrightarrow{pp_1}\|e_{x_1} & y_1 - y + \|\overrightarrow{pp_1}\|e_{y_1} & z_1 - z + \|\overrightarrow{pp_1}\|e_{z_1} \\ x_2 - x + \|\overrightarrow{pp_2}\|e_{x_2} & y_2 - y + \|\overrightarrow{pp_2}\|e_{y_2} & z_2 - z + \|\overrightarrow{pp_2}\|e_{z_2} \\ x_3 - x + \|\overrightarrow{pp_3}\|e_{x_3} & y_3 - y + \|\overrightarrow{pp_3}\|e_{y_3} & z_3 - z + \|\overrightarrow{pp_3}\|e_{z_3} \\ x_4 - x + \|\overrightarrow{pp_4}\|e_{x_4} & y_4 - y + \|\overrightarrow{pp_4}\|e_{y_4} & z_4 - z + \|\overrightarrow{pp_4}\|e_{z_4} \end{bmatrix} \mathbf{S}^T$$

$$= \mathbf{XS}^T \qquad (\textit{inconsistent}) \tag{B4}$$

$$\mathbf{Y} := \mathbf{Y}_1 - \mathbf{Y}_2 \quad \mathbf{X} := \mathbf{Y}_1 + \mathbf{Y}_2 \tag{B5}$$

$$\mathbf{y} := \text{vec}\mathbf{Y} = (\mathbf{I}_3 \otimes \mathbf{X})\text{vec}\mathbf{S}^T + \mathbf{v} = (\mathbf{I}_3 \otimes \mathbf{X})\mathbf{K}\text{veck } \mathbf{S}^T + \mathbf{v} \tag{B6}$$

$$\mathbf{A} := (\mathbf{I}_3 \otimes \mathbf{X})\mathbf{K} \quad \mathbf{u} := \text{veck } \mathbf{S}^T = [a\ b\ c]^T \tag{B7}$$

$$\mathbf{y} = \mathbf{Au} + \mathbf{v} \tag{B8}$$

$$\|\mathbf{v}\|^2 = \min_{\mathbf{u}} \Longleftrightarrow \hat{\mathbf{u}} = [\hat{a}\ \hat{b}\ \hat{c}]^T = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \tag{B9}$$

*Box B.2:* Transformation between observed coordinates of the relative placement vector $\mathbf{x}_i - \mathbf{x}$ in a local frame and relative Cartesian coordinates in a global reference frame, *case(ii)*: image coordinates $\{\bar{x}_i, \bar{y}_i, -f\}$ in a *local camera/CCD sensor frame.*

"same as (B1)"

$$\tan\alpha_i = \bar{y}_i/\bar{x}_i \quad \tan\beta_i = -f/\sqrt{\bar{x}_i^2 + \bar{y}_i^2} \tag{B10i}$$

$$\cos\alpha_i = \frac{1}{\sqrt{1+\tan^2\alpha_i}} = \frac{\bar{x}_i}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2}}$$
$$\cos\beta_i = \frac{1}{\sqrt{1+\tan^2\beta_i}} = \frac{\sqrt{\bar{x}_i^2 + \bar{y}_i^2}}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + f^2}} \tag{B10ii}$$

$$\sin\alpha_i = \frac{\tan\alpha_i}{\sqrt{1+\tan^2\alpha_i}} = \frac{\bar{y}_i}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2}} \quad \sin\beta_i = \frac{\tan\beta_i}{\sqrt{1+\tan^2\beta_i}} = -\frac{f}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + f^2}} \tag{B10iii}$$

"for details consult *E. Grafarend, P. Lohse and B. Shaffrin* (1989, pp. 172–175)"

$$\begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix} = \mathbf{R}\frac{\|\overrightarrow{pp_i}\|}{\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + f^2}} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ -f \end{bmatrix} = \mathbf{R}\frac{\|\overrightarrow{pp_i}\|}{\bar{d}_i} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix} \qquad \forall\ i \in \{1,2,3,4\} \tag{B11}$$

subject to

$$\sqrt{\bar{x}_i^2 + \bar{y}_i^2 + f^2} := \bar{d}_i \qquad \bar{z}_i = -f$$

*Continued*

(*Box B.2 continued*)

"same as (B.2) and (B.3)"

$$\mathbf{Y} := \begin{bmatrix} x_1 - x - \|\overrightarrow{pp_1}\|\bar{x}_1/\bar{d}_1 & y_1 - y - \|\overrightarrow{pp_1}\|\bar{y}_1/\bar{d}_1 & z_1 - z - \|\overrightarrow{pp_1}\|\bar{z}_1/\bar{d}_1 \\ x_2 - x - \|\overrightarrow{pp_2}\|\bar{x}_2/\bar{d}_2 & y_2 - y - \|\overrightarrow{pp_2}\|\bar{y}_2/\bar{d}_2 & z_2 - z - \|\overrightarrow{pp_2}\|\bar{z}_2/\bar{d}_2 \\ x_3 - x - \|\overrightarrow{pp_3}\|\bar{x}_3/\bar{d}_3 & y_3 - y - \|\overrightarrow{pp_3}\|\bar{y}_3/\bar{d}_3 & z_3 - z - \|\overrightarrow{pp_3}\|\bar{z}_3/\bar{d}_3 \\ x_4 - x - \|\overrightarrow{pp_4}\|\bar{x}_4/\bar{d}_4 & y_4 - y - \|\overrightarrow{pp_4}\|\bar{y}_4/\bar{d}_4 & z_4 - z - \|\overrightarrow{pp_4}\|\bar{z}_4/\bar{d}_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x + \|\overrightarrow{pp_1}\|\bar{x}_1/\bar{d}_1 & y_1 - y + \|\overrightarrow{pp_1}\|\bar{y}_1/\bar{d}_1 & z_1 - z + \|\overrightarrow{pp_1}\|\bar{z}_1/\bar{d}_1 \\ x_2 - x + \|\overrightarrow{pp_2}\|\bar{x}_2/\bar{d}_2 & y_2 - y + \|\overrightarrow{pp_2}\|\bar{y}_2/\bar{d}_2 & z_2 - z + \|\overrightarrow{pp_2}\|\bar{z}_2/\bar{d}_2 \\ x_3 - x + \|\overrightarrow{pp_3}\|\bar{x}_3/\bar{d}_3 & y_3 - y + \|\overrightarrow{pp_3}\|\bar{y}_3/\bar{d}_3 & z_3 - z + \|\overrightarrow{pp_3}\|\bar{z}_3/\bar{d}_3 \\ x_4 - x + \|\overrightarrow{pp_4}\|\bar{x}_4/\bar{d}_4 & y_4 - y + \|\overrightarrow{pp_4}\|\bar{y}_4/\bar{d}_4 & z_4 - z + \|\overrightarrow{pp_4}\|\bar{z}_4/\bar{d}_4 \end{bmatrix} \mathbf{S}^T$$

$$= \mathbf{XS}^T \qquad (\textit{inconsistent}) \tag{B12}$$

"same as (B5) – (B9)"

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