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Closed form solution to the twin P4P or the combined three dimensional resection-intersection problem in terms of Möbius barycentric coordinates

E. Grafarend , J. Shan

Department of Geodetic Science, Stuttgart University, Keplerstrasse 11, D-70174 Stuttgart, Germany

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Abstract. The *twin perspective 4 point* (twin P4P) problem – also called the combined three dimensional resection-intersection problem – is the problem of finding the position of a scene object from 4 correspondence points and a scene stereopair. While the *perspective centers* of the left and right scene image are positioned by means of a *double three dimensional resection*, the position of the scene object imaged on the left and right photograph is determined by a *three dimensional intersection* based upon given *resected perspective centers*. Here we present a new *algorithm* solving the *twin P4P* problem by means of *Möbius barycentric coordinates*. In the *first algorithmic step* we determine the distances between the perspective centers and the unknown intersected point by solving a linear system of equations. Typically, area elements of the left and right image build up the linear equation system. The *second algorithmic step* allows for the computation of the *Möbius barycentric coordinates* of the *unknown intersected point* which are *thirdly* converted into three dimensional object space coordinates $\{X, Y, Z\}$ of the intersected point. Typically, this *three-step algorithm* based upon *Möbius barycentric coordinates* takes advantage of the primary double resection problem from which *only distances* from four correspondence points to the left and right perspective centre are needed. No orientation parameters and no coordinates of the left and right perspective center have to be made available.

Introduction

The *twin perspective 4 point* (twin P4P) problem - also called the *combined three dimensional resection-intersection problem* (*Figure 1.1*) – *is the problem of finding the positions of a scene object from 4 correspondence points and a left/right scene image* (*stereoscopic machine or computer vision*). *While the perspective centers* of the left and right scene image are positioned by means of a *double three dimensional resection*, the positions of a scene object imaged on the left and right are determined by a *three dimensional intersection* based upon resected perspective centers.

Our new algorithmic solution to the *twin P4P* problem will be based on *Möbius barycentric coordinates* (A.F. Möbius,1827) which are reviewed by M. Berger (1994, p.67-82), for instance, and are applied by W. Pachelski and E. Grafarend (1994) and W. Pachelski(1994). As an alternative solution of P4P we refer to M.A. Abidi and T. Chandra (1995). *Section 1* is an introduction of *Möbius barycentric coordinates* illustrated by Figure 1.2 with respect to an *affine basis* generated by the simplex of 4 correspondence points.

Section 2 aims at a constructive setup of *three algorithmic steps* to solve the *twin P4P*. At first we develop the perspective equations of P4P before secondly we represent the *volume coordinates* Δ_i of barycentric type in terms of the left and right *perspective area elements*. With respect to *volume coordinates* Δ_i we suceed to setup the *first algorithmic step*, namely the computation of distances *from* the left *and* right perspective center to the intersected point by solving a linear system of equations. As an *input* we need only (*i*) observed image coordinates of the 4 correspondence points and of the one intersected point, (*ii*) perspective area elements of type left and right and (*iii*) distances *from* the left and right perspective center to the four correspondence points as produced from the first step of solving the three dimensional double resection also called the *Grunert equations* (J.A. Grunert (1841), E. Grafarend, P. Lohse and B. Schaffrin (1989), E. Grafarend and J. Shan (1996), R.M. Haralick et al (1994), F. Müller (1925)). As soon as the distances in the seven dimensional simplex are determined, by means of the *second algorithmic step*, we are able to compute the four barycentric coordinates $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ of the *intersected point* from derived volume coordinates Δ_i. Finally by the *third algorithmic step*, the barycentric coordinates $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ are converted into

Correspondence to: E. Grafarend

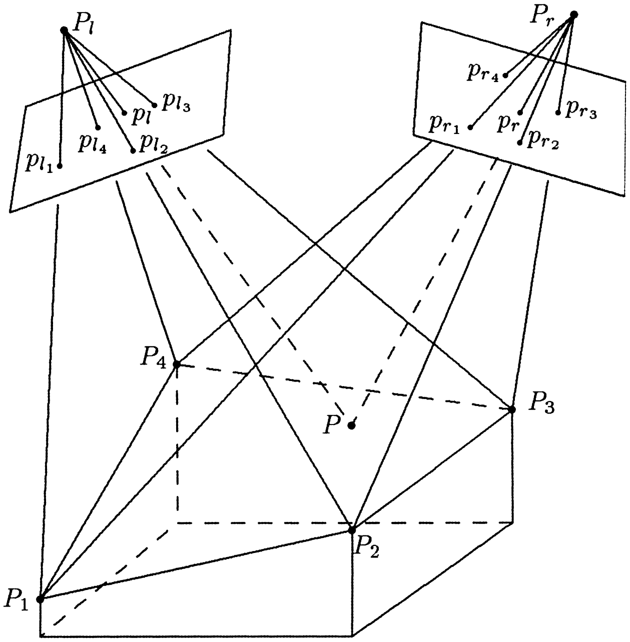

Fig. 1.1 Twin P4P or double resection - intersection problem

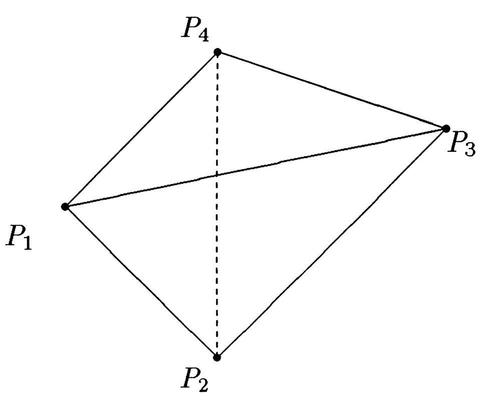

Fig. 1.2 Affine basis based upon the four dimensional simplex $\{P_1, P_2, P_3, P_4\}$ in $\mathscr{R}^3$

Cartesian coordinates $\{X, Y, Z\}$ of the *intersected point* with respect to the Cartesian coordinates of the four correspondence points.

Section 3 focuses on a discussion of the three step algorithm solving *twin P4P* by means of *Möbius barycentric coordinates*, in particular on those transformations which leave the barycentric observations of *volume coordinates* and *perspective area elements* invariant. A numeric example with an aerial stereopair is presented to depict our algorithm

1. Setup of twin P4P and the Möbius barycentric coordinates

The primary situation of the *twin P4P* in the three dimensional Euclidean manifold of standard metric $\boldsymbol{E}^3 := \{\mathscr{R}^3, g_{\mu\nu}\}$ subject to $\mu, \nu \in \{1,2,3\}$ is as follows: By means of an *intersection problem*, directions to an unknown point $P \in \boldsymbol{E}^3$ are observed from the *left perspective center* P_l as well as the *right perspective center* P_r. In turn, the left and right perspective center are positioned by direction observations to at least *four known points* $P_i \in \boldsymbol{E}^3$ subject to $i \in \{1,2,3,4\}$ by means of a *twin resection* also called "double resection" or "binary resection". *Figure 1.1* illustrates the graph of the seven dimensional simplex constituted by (*i*) the intersected unknown point P, (*ii*) the resected perspective center P_l and P_r and (*iii*) four given points $P_i \in \mathbf{E}^3$. Those four known points generate an affine basis (A.F. Möbius (1827), *A. Berger* (1994), pp. 67-82), namely.

$$\{\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}, \overrightarrow{P_1P_4}\} \; or \; \{\boldsymbol{X}_2 - \boldsymbol{X}_1, \boldsymbol{X}_3 - \boldsymbol{X}_1, \boldsymbol{X}_4 - \boldsymbol{X}_1\}$$

where $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4\}$ are *placement vectors*. Equivalently, they span an $\mathscr{R}^3$ equipped with a general metric $g_{\mu\nu}$. With respect to the *tetrahedron* $\{P_1, P_2, P_3, P_4\}$, *Figure 1.2* is a visualization of the *affine basis* subject to *affine geometry*. Note that an *affine basis* is defined as a basis of an $\mathscr{R}^3$ which is *translational invariant* or *equivalent under the action of the translation group*. In addition, in the definition of the *affine basis* we have used the *equivalence relation* $P_i \sim \boldsymbol{X}_i$ (i = 1, 2, 3, 4).

The point $P \sim \boldsymbol{X}$ can be represented in the *affine basis* by (1.1)-(1.5) of *Box 1.1* where $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ are the *Möbius barycentric coordinates* of $P \sim \boldsymbol{X}$. In particular we have introduced $\Lambda_1 := 1 - (\Lambda_2 + \Lambda_3 + \Lambda_4)$. As soon as we cover $\boldsymbol{E}^3$ by *Cartesian coordinates* $\{X, Y, Z\}$ in one global chart, via (1.4) we are led *from* (1.3) to (1.5). $\{X, Y, Z, 1\}$ are called *homogeneous coordinates* of P as motivated in E. Grafarend and J. Shan (1996), for instance.

If the *Cartesian coordinates* of the point $P \sim \boldsymbol{X}$ are given, its *Möbius barycentric coordinates* $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ with respect to the *affine basis* can be computed via (1.6)-(1.12). The corresponding determinants $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta\}$ characterize six times the *volume of the respective tetrahedra* $\{P, P_2, P_3, P_4\}$, $\{P_1, P, P_3, P_4\}, \cdots \{P_1, P_2, P_3, P_4\}$. It is for this reason that the *quadruple coordinates* $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ of a point $P \in \mathscr{R}^3$, namely the *Möbius barycentric coordinates*, are called *volume coordinates*, too.

Box 1.1. Möbius barycentric coordinates, affine basis

$$\boldsymbol{X} - \boldsymbol{X}_1 = (\boldsymbol{X}_2 - \boldsymbol{X}_1)\Lambda_2 + (\boldsymbol{X}_3 - \boldsymbol{X}_1)\Lambda_3 + (\boldsymbol{X}_4 - \boldsymbol{X}_1)\Lambda_4 \tag{1.1}$$

$$\boldsymbol{X} = \boldsymbol{X}_1(1 - \Lambda_2 - \Lambda_3 - \Lambda_4) + \boldsymbol{X}_2\Lambda_2 + \boldsymbol{X}_3\Lambda_3 + \boldsymbol{X}_4\Lambda_4 \tag{1.2}$$

$$\boldsymbol{X} = \boldsymbol{X}_1\Lambda_1 + \boldsymbol{X}_2\Lambda_2 + \boldsymbol{X}_3\Lambda_3 + \boldsymbol{X}_4\Lambda_4 \tag{1.3}$$

subject to

Continued

(Box 1.1. Continued)

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 = 1 \tag{1.4}$$

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix} \tag{1.5}$$

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \tag{1.6}$$

$$\Lambda_i = \frac{\Delta_i}{\Delta} \qquad \forall_i \in \{1,2,3,4\} \tag{1.7}$$

$$\Delta_1 := \begin{vmatrix} X & X_2 & X_3 & X_4 \\ Y & Y_2 & Y_3 & Y_4 \\ Z & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6\mathrm{vol}\{P, P_2, P_3, P_4\} \tag{1.8}$$

$$\Delta_2 := \begin{vmatrix} X_1 & X & X_3 & X_4 \\ Y_1 & Y & Y_3 & Y_4 \\ Z_1 & Z & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6\mathrm{vol}\{P_1, P, P_3, P_4\} \tag{1.9}$$

$$\Delta_3 := \begin{vmatrix} X_1 & X_2 & X & X_4 \\ Y_1 & Y_2 & Y & Y_4 \\ Z_1 & Z_2 & Z & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6\mathrm{vol}\{P_1, P_2, P, P_4\} \tag{1.10}$$

$$\Delta_4 := \begin{vmatrix} X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ Z_1 & Z_2 & Z_3 & Z \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6\mathrm{vol}\{P_1, P_2, P_3, P\} \tag{1.11}$$

$$\Delta := \begin{vmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 6\mathrm{vol}\{P_1, P_2, P_3, P_4\} \tag{1.12}$$

2. Solving twin P4P by means of Möbius barycentric coordinates

At first let us assume that by means of a solution to the *double resection problem* based on *Möbius barycentric coordinates* – see *E. Grafarend and J. Shan* (1996, equations (2.8)-(2.10) subject to (3.1)-(3.9)) – we have determined the Cartesian coordinates $\{X_l, Y_l, Z_l\}$ and $\{X_r, Y_r, Z_r\}$, respectively, of the *left perspective center* P_l and the *right perspective center* P_r, respectively, illustrated in *Figure 1.1.*

Box 2.1. The perspective equations of twin P4P.

$$\begin{bmatrix} X - X_l \\ Y - Y_l \\ Z - Z_l \end{bmatrix} = s_l \boldsymbol{R}_l \begin{bmatrix} x_l \\ y_l \\ -f_l \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} X - X_r \\ Y - Y_r \\ Z - Z_r \end{bmatrix} = s_r \boldsymbol{R}_r \begin{bmatrix} x_r \\ y_r \\ -f_r \end{bmatrix} \tag{2.2}$$

subject to

$$s_l := \frac{\|\overrightarrow{PP_l}\|}{\|\overrightarrow{pp_l}\|} = \frac{\sqrt{(X - X_l)^2 + (Y - Y_l)^2 + (Z - Z_l)^2}}{\sqrt{x_l^2 + y_l^2 + f_l^2}} \tag{2.3}$$

$$s_r := \frac{\|\overrightarrow{PP_r}\|}{\|\overrightarrow{pp_r}\|} = \frac{\sqrt{(X - X_r)^2 + (Y - Y_r)^2 + (Z - Z_r)^2}}{\sqrt{x_r^2 + y_r^2 + f_r^2}} \tag{2.4}$$

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= s_l \boldsymbol{R}_l \begin{bmatrix} x_l \\ y_l \\ -f_l \end{bmatrix} + \begin{bmatrix} X_l \\ Y_l \\ Z_l \end{bmatrix} \\ &= s_r \boldsymbol{R}_r \begin{bmatrix} x_r \\ y_r \\ -f_r \end{bmatrix} + \begin{bmatrix} X_r \\ Y_r \\ Z_r \end{bmatrix} \end{aligned} \tag{2.5}$$

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} &= s_l \left|\begin{array}{c|c} \boldsymbol{R}_l & \begin{matrix} X_l \\ Y_l \\ Z_l \end{matrix} \\ \hline \boldsymbol{0} & 1 \end{array}\right| \begin{bmatrix} x_l \\ y_l \\ -f_l \\ s_l^{-1} \end{bmatrix} \\ &= s_r \left|\begin{array}{c|c} \boldsymbol{R}_r & \begin{matrix} X_r \\ Y_r \\ Z_r \end{matrix} \\ \hline \boldsymbol{0} & 1 \end{array}\right| \begin{bmatrix} x_r \\ y_r \\ -f_r \\ s_r^{-1} \end{bmatrix} \end{aligned} \tag{2.6}$$

In addition, let us denote by $\{X - X_l, Y - Y_l, Z - Z_l\}$ and $\{X - X_r, Y - Y_r, Z - Z_r\}$ the difference *Cartesian coordinate* of the unknown point P which is to be intersected from the *left perspective center* P_l as well as from the *right perspective center* P_r. Following *perspective geometry* the difference Cartesian coordinates $\{X - X_l, Y - Y_l, Z - Z_l\}$ and $\{X - X_r, Y - Y_r, Z - Z_r\}$, respectively via (2.1), (2.2), can be represented in terms of *left and right image coordinates* $\{x_l, y_l, -f_l\}$ and $\{x_r, y_r, -f_r\}$ of the point P with respect to the *left image plane*/left photograph frame and the *right image plane*/right photgraph frame. f_l, f_r respectively denote the *left* *focal length, the right focal length, respectively.* s_l, s_r, *and* $\boldsymbol{R}_l, \boldsymbol{R}_r$, respectively refer to the *left, right, scale ratio* and to the *left, right three dimensional rotation matrix*, elements of the special orthogonal group SO(3) *in three dimensions*.

Box 2.2. Möbius barycentric coordinates of point P intersected from left and right perspective center (*taken* Δ_1 *as an example*)

$$\Delta_1 = \begin{vmatrix} X & X_2 & X_3 & X_4 \\ Y & Y_2 & Y_3 & Y_4 \\ Z & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= s_l s_{l_2} s_{l_3} s_{l_4} \left| \begin{array}{c|c} \boldsymbol{R}_l & \begin{array}{c} X_l \\ Y_l \\ Z_l \end{array} \\ \hline \boldsymbol{0} & 1 \end{array} \right| \begin{vmatrix} x_l & x_{l_2} & x_{l_3} & x_{l_4} \\ y_l & y_{l_2} & y_{l_3} & y_{l_4} \\ -f_l & -f_l & -f_l & -f_l \\ s_l^{-1} & s_{l_2}^{-1} & s_{l_3}^{-1} & s_{l_4}^{-1} \end{vmatrix}$$

$$= s_r s_{r_2} s_{r_3} s_{r_4} \left| \begin{array}{c|c} \boldsymbol{R}_r & \begin{array}{c} X_r \\ Y_r \\ Z_r \end{array} \\ \hline \boldsymbol{0} & 1 \end{array} \right| \begin{vmatrix} x_r & x_{r_2} & x_{r_3} & x_{r_4} \\ y_r & y_{r_2} & y_{r_3} & y_{r_4} \\ -f_r & -f_r & -f_r & -f_r \\ s_r^{-1} & s_{r_2}^{-1} & s_{r_3}^{-1} & s_{r_4}^{-1} \end{vmatrix} \quad (2.7)$$

subject to

$$S_{l_2} := \frac{\| \overrightarrow{P_2 P_l} \|}{\| \overrightarrow{p_2 p_l} \|} = \frac{\sqrt{(X_2 - X_l)^2 + (Y_2 - Y_l)^2 + (Z_2 - Z_l)^2}}{\sqrt{x_{l_2}^2 + y_{l_2}^2 + f_l^2}}, \cdots$$

$$S_{r_4} := \frac{\| \overrightarrow{P_4 P_r} \|}{\| \overrightarrow{p_4 pr} \|} = \frac{\sqrt{(X_4 - X_r)^2 + (Y_4 - Y_r)^2 + (Z_4 - Z_r)^2}}{\sqrt{x_{r_4}^2 + y_{r_4}^2 + f_r^2}} \quad (2.8)$$

$$\left| \begin{array}{c|c} \boldsymbol{R}_l & \begin{array}{c} X_l \\ Y_l \\ Z_l \end{array} \\ \hline \boldsymbol{0} & 1 \end{array} \right| = |\boldsymbol{R}_l| = 1 \quad (2.9_l)$$

$$\left| \begin{array}{c|c} \boldsymbol{R}_r & \begin{array}{c} X_r \\ Y_r \\ Z_r \end{array} \\ \hline \boldsymbol{0} & 1 \end{array} \right| = |\boldsymbol{R}_l| = 1 \quad (2.9_l)$$

For derivation of (2.1), (2.2) we refer to *E. Grafarend* (1983, formulae 1(1)-1(6)), *for instance*. (2.3),(2.4) are left, right length ratios being expressed with respect to Euclidean norms of $\overrightarrow{PP_l}$, $\overrightarrow{PP_r}$, $\overrightarrow{pp_l}, \overrightarrow{pp_r}$.

Box 2.3. Representation of the volume coordinates in terms of left and right perspective area elements (*taken* Δ_1 *and* Δ_2 *as example*)

$$\Delta_1 = \begin{vmatrix} X & X_2 & X_3 & X_4 \\ Y & Y_2 & Y_3 & Y_4 \\ Z & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = s_l s_{l_2} s_{l_3} s_{l_4}$$

$$\times \left\{ s_l^{-1} f_l \begin{vmatrix} x_{l_2} & x_{l_3} & x_{l_4} \\ y_{l_2} & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} - s_{l_2}^{-1} f_l \begin{vmatrix} x_l & x_{l_3} & x_{l_4} \\ y_l & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} \right.$$

$$\left. + s_{l_3}^{-1} f_l \begin{vmatrix} x_l & x_{l_2} & x_{l_4} \\ y_l & y_{l_2} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} - s_{l_4}^{-1} f_l \begin{vmatrix} x_l & x_{l_2} & x_{l_3} \\ y_l & y_{l_2} & y_{l_3} \\ 1 & 1 & 1 \end{vmatrix} \right\} \quad (2.10_l)$$

$$\Delta_1 = \begin{vmatrix} X & X_2 & X_3 & X_4 \\ Y & Y_2 & Y_3 & Y_4 \\ Z & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = s_r s_{r_2} s_{r_3} s_{r_4}$$

$$\times \left\{ s_r^{-1} f_r \begin{vmatrix} x_{r_2} & x_{r_3} & x_{r_4} \\ y_{r_2} & y_{r_3} & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} - s_{r_2}^{-1} f_r \begin{vmatrix} x_r & x_{r_3} & x_{r_4} \\ y_r & y_{r_3} & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} \right.$$

$$\left. + s_{r_3}^{-1} f_r \begin{vmatrix} x_r & x_{r_2} & x_{r_4} \\ y_r & y_{r_2} & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} - s_{r_4}^{-1} f_r \begin{vmatrix} x_r & x_{r_2} & x_{r_3} \\ y_r & y_{r_2} & y_{r_3} \\ 1 & 1 & 1 \end{vmatrix} \right\} \quad (2.10_r)$$

"*example*":

$$\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ -f & -f & -f \end{vmatrix} = -f \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ 1 & 1 & 1 \end{vmatrix} = -2f area\{p_2, p_3, p_4\}$$

$$\begin{aligned} \Delta_1(left) = {} & 2 f_l s_{l_2} s_{l_3} s_{l_4}\ area_l\{p_2, p_3, p_4\} \\ & + 2 f_l[-s_{l_3} s_{l_4}\ area_l\{p, p_3, p_4\} \\ & + s_{l_2} s_{l_4}\ area_l\{p, p_2, p_4\} \\ & - s_{l_2} s_{l_3}\ area_l\{p, p_2, p_3\}] s_l \end{aligned} \quad (2.11_l)$$

Continued

(*Box 2.3. Continued*)

$$\Delta_1(right) = 2f_r s_{r_2} s_{r_3} s_{r_4}\ area_r\{p_2, p_3, p_4\} + 2f_r[-s_{r_3} s_{r_4}\ area_r\{p, p_3, p_4\} + s_{r_2} s_{r_4}\ area_r\{p, p_2, p_4\} - s_{r_2} s_{r_3}\ area_r\{p, p_2, p_3\}]s_r \quad (2.11_r)$$

$$\Delta_1(left) = \Delta_1(right) \quad (2.12)$$

similarly for Δ_2:

$$\Delta_2 = \begin{vmatrix} X_1 & X & X_3 & X_4 \\ Y_1 & Y & Y_3 & Y_4 \\ Z_1 & Z & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = s_{l_1} s_l s_{l_3} s_{l_4}$$

$$\times \left\{ s_{l_1}^{-1} f_l \begin{vmatrix} x_l & x_{l_3} & x_{l_4} \\ y_l & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} - s_l^{-1} f_l \begin{vmatrix} x_{l_1} & x_{l_3} & x_{l_4} \\ y_{l_1} & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} + s_{l_3}^{-1} f_l \begin{vmatrix} x_{l_1} & x_l & x_{l_4} \\ y_{l_1} & y_l & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} - s_{l_4}^{-1} f_l \begin{vmatrix} x_{l_1} & x_l & x_{l_3} \\ y_{l_1} & y_l & y_{l_3} \\ 1 & 1 & 1 \end{vmatrix} \right\} \quad (2.13_l)$$

$$\Delta_2 = \begin{vmatrix} X_1 & X & X_3 & X_4 \\ Y_1 & Y & Y_3 & Y_4 \\ Z_1 & Z & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = s_{r_1} s_r s_{r_3} s_{r_4}$$

$$\times \left\{ s_{r_1}^{-1} f_r \begin{vmatrix} x_r & x_{r_3} & x_{r_4} \\ y_r & y_{r_3} & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} - s_r^{-1} f_r \begin{vmatrix} x_{r_1} & x_{r_3} & x_{r_4} \\ y_{r_1} & y_{r_3} & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} + s_{r_3}^{-1} f_r \begin{vmatrix} x_{r_1} & x_r & x_{r_4} \\ y_{r_1} & y_r & y_{r_4} \\ 1 & 1 & 1 \end{vmatrix} - s_r^{-1} f_r \begin{vmatrix} x_{r_1} & x_r & x_{r_3} \\ y_{r_1} & y_r & y_{r_3} \\ 1 & 1 & 1 \end{vmatrix} \right\} \quad (2.13_r)$$

$$\Delta_2(left) = 2f_l s_{l_1} s_{l_3} s_{l_4}\ area_l\{p_1, p_3, p_4\} + 2f_l[-s_{l_3} s_{l_4}\ area_l\{p, p_3, p_4\} + s_{l_1} s_{l_4}\ area_l\{p_1, p, p_4\} - s_{l_1} s_{l_3}\ area_l\{p_1, p, p_3\}]s_l \quad (2.14_l)$$

$$\Delta_2(right) = 2f_r s_{r_1} s_{r_3} s_{r_4}\ area_r\{p_1, p_3, p_4\} + 2f_r[-s_{r_3} s_{r_4}\ area_r\{p, p_3, p_4\} + s_{r_1} s_{r_4}\ area_r\{p_1, p, p_4\} - s_{r_1} s_{r_3} area_r\{p_1, p, p_3\}]s_r \quad (2.14_r)$$

$$\Delta_2(left) = \Delta_2(right) \quad (2.15)$$

Once we solve (2.1), (2.2) for the Cartesian coordinates $\{X, Y, Z\}$ of the intersected point P we receive the standard representation of the transformation from "*image coordinates*" $\{x_l, y_l, -f_l\}$, $\{x_r, y_r, -f_r\}$ into "*object space coordinates*": There appear left, right *scale ratio*, s_l, s_r, left, right *rotation* $\boldsymbol{R}_l, \boldsymbol{R}_r$ both elements of SO(3) and left right *translation* $\{X_l, Y_l, Z_l\}, \{X_r, Y_r, Z_r\}$· (2.6) is the counterpart of (2.5) in the form of homogeneous coordinates.

Secondly, we shall transform the *unknown Cartesian homogenous coordinates* (2.6) of the intersected point P by means of (1.6) into *Möbius barycentric coordinates* Λ_i, namely Δ_i (2.7). The left, right *image coordinates* of the given points $\{P_1, P_2, P_3, P_4\}$ are denoted by $\{x_{l_i}, y_{l_i}, -f_l\}, \{x_{r_i}, y_{r_i}, -f_r\}, i \in \{1, 2, 3, 4\}$, respectively. The expansion of the left and right determinant of the *volume coordinates* Δ_1, Δ_2, e.g., leads us to the decomposition (2.10)-(2.15) in terms of *area elements of type left and right image*. In the equations (2.12) and (2.15) only the length ratios s_l and s_r are unknown. As collected in *Box 2.4* they can be used to establish a system of *two linear equations* in two unknowns s_l and s_r. The inhomogeneity of the system of linear equations is given by (2.18),(2.19), while the coefficient matrix $\boldsymbol{A}$ by (2.20)-(2.23). We could have inverted $\mathbf{x} = \boldsymbol{A}^{-1}\mathbf{y}$ the system of inhomogenous linear equation $\mathbf{y} = \boldsymbol{A}\mathbf{x}$ in order to obtain s_l and s_r, but we prefer the solution from the numerical computation.

Box 2.4. Determination of left and right length ratios s_l and s_r by means of a system of two linear equations.

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 = y_1 \\ a_{21}x_1 + a_{22}x_2 = y_2 \end{matrix} \sim \mathbf{Ax} = \mathbf{y} \quad (2.16)$$

$$x_1 := s_l, \quad x_2 := s_r \quad (2.17)$$

$$y_1 := s_{l_2} s_{l_3} s_{l_4}\ area_l\{p_2, p_3, p_4\} - s_{r_2} s_{r}3 s_{r_4}\ area_r\{p_2, p_3, p_4\} \quad (2.18)$$

$$y_2 := s_{l_1} s_{l_3} s_{l_4} area_l\{p_1, p_3, p_4\} - s_{r_1} s_{r_3} s_{r_4} area_r\{p_1, p_3, p_4\} \quad (2.19)$$

$$a_{11} := s_{l_2} s_{l_3} s_{l_4} \times [s_{l_2}^{-1} area_l\{p, p_3, p_4\} - s_{l_3}^{-1} area_l\{p, p_2, p_4\} + s_{l_4}^{-1} area_l\{p, p_2, p_3\}] \quad (2.20)$$

$$a_{12} := -s_{r_2} s_{r_3} s_{r_4} \times [s_{r_2}^{-1} area_r\{p, p_3, p_4\} - s_{r_3}^{-1} area_r\{p, p_2, p_4\} + s_{r_4}^{-1} area_r\{p, p_2, p_3\}] \quad (2.21)$$

Continued

(*Box 2.4 Continued*)

$$a_{21} := s_{l_1}s_{l_3}s_{l_4} \times [s_{l_1}^{-1} area_l\{p, p_3, p_4\} + s_{l_3}^{-1} area_l\{p_1, p, p_4\} - s_{l_4}^{-1} area_l\{p_1, p, p_3\}] \quad (2.22)$$

$$a_{22} := -s_{r_1}s_{r_3}s_{r_4} \times [s_{r_1}^{-1} area_r\{p, p_3, p_4\} + s_{r_3}^{-1} area_r\{p_1, p_3, p_4\} - s_{r_4}^{-1} area_r\{p_1, , p, p_4\}] \quad (2.23)$$

On the basis of *left and right scale ratio* we compute the *Möbius barycentric coordinates* by means of (1.7)-(1.12) and (2.7), (2.13) etc., as outlined in *Box 2.5*. Those barycentric coordinates $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ have been given as *determinantal ratios* n (2.24)-(2.27), where we had not to specify "*left*" or "*right*". Any choice can be made. For instance, if we prefer a computation based upon coordinates in the *left image plane*, we can use (2.24)-(2.27) *in the left mode*. All quantities involved have to be indexed "left".

Box 2.5. Determination of Möbius barycentric coordinates $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ of the unknown point which is intersected

$$\Lambda_1 = \frac{\Delta_1}{\Delta} = \frac{s\begin{vmatrix} x & x_2 & x_3 & x_4 \\ y & y_2 & y_3 & y_4 \\ -f & -f & -f & -f \\ s^{-1} & s_2^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}}{s_1\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}} \quad (2.24)$$

$$\Lambda_2 = \frac{\Delta_2}{\Delta} = \frac{s\begin{vmatrix} x_1 & x & x_3 & x_4 \\ y_1 & y & y_3 & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}}{s_2\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}} \quad (2.25)$$

$$\Lambda_3 = \frac{\Delta_3}{\Delta} = \frac{s\begin{vmatrix} x_1 & x_2 & x & x_4 \\ y_1 & y_2 & y & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s^{-1} & s_4^{-1} \end{vmatrix}}{s_3\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}} \quad (2.26)$$

(*Box 2.5 Continued*)

$$\Lambda_4 = \frac{\Delta_4}{\Delta} = \frac{s\begin{vmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s_3^{-1} & s^{-1} \end{vmatrix}}{s_4\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ -f & -f & -f & -f \\ s_1^{-1} & s_2^{-1} & s_3^{-1} & s_4^{-1} \end{vmatrix}} \quad (2.27)$$

Thirdly with given Möbius barycentric coordinates $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ which are functions of the image coordinates of the four reference points $\{x_i, y_i, -f\}, i \in 1, 2, 3, 4$ *either left or right and* the image coordinates $\{x, y, -f\}$ again *either left or right*, of the intersected *unknown point*, by means of (1.5) we compute the *homogeneous coordinates* $\{X, Y, Z, 1\}$ of the unknown point P.

In addition, we reflect the functional influence of the scale ratios s_1, s_2, s_3, s_4 and s, either of *left or right* type, in the representation of *Möbius barycentric coordinates* (2.24)-(2.27). The transformation (1.5) $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}, s_1, s_2, s_3, s_4, s)$ with respect to the base $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4\}$ has not been explicitly written since it is beneficial to concentrate instead on the numerical computation.

Box 2.6. Transformation of horizontal - vertical directions $\{\alpha, \beta\}$ of a theodolite into image coordinates $\{x, y, -f\}$.

$$tan\alpha_l = \frac{y_l}{x_l} \quad tan\beta_l = -\frac{f_l}{\sqrt{x_l^2 + y_l^2}} \quad (2.28)$$

$$tan\alpha_r = \frac{y_r}{x_r} \quad tan\beta_r = -\frac{f_r}{\sqrt{x_r^2 + y_r^2}} \quad (2.29)$$

$$\begin{aligned} x_l &= \sqrt{x_l^2 + y_l^2 + f_l^2} \cos\alpha_l \cos\beta_l \\ &= -f_l \cot\beta_l \cos\alpha_l \\ y_l &= \sqrt{x_l^2 + y_l^2 + f_l^2} \sin\alpha_l \cos\beta_l \\ &= -f_l \cot\beta_l \sin\alpha_l \\ -f_l &= \sqrt{x_l^2 + y_l^2 + f_l^2} \sin\beta_l \\ x_r &= \sqrt{x_r^2 + y_r^2 + f_r^2} \cos\alpha_r \cos\beta_r \\ &= -f_r \cot\beta_r \cos\alpha_r \end{aligned} \quad (2.30)$$

Continued

(*Box 2.6. Continued*)

$$y_r = \sqrt{x_r^2 + y_r^2 + f_r^2}\sin\alpha_r\cos\beta_r \quad (2.31)$$
$$= -f_r\cot\beta_r\sin\alpha_r$$
$$-f_r = \sqrt{x_r^2 + y_r^2 + f_r^2}\sin\beta_r$$

This algorithm could be easily extended to geodetic application where the observations of *image coordinates* $\{x_l, y_l, -f_l\}$ and $\{x_r, y_r, -f_r\}$ of the unknown point P are not available, but rather its *theodolitic observations* of *left horizontal directions, left vertical directions* $\{\alpha_l, \beta_l\}$, as well as *right horizontal directions, right vertical directions* $\{\alpha_r, \beta_r\}$. In this case, we have to replace all *image coordinates* in all formulae by the transformation outlined in *Box 2.6*.

It should be noted that f_l, f_r could be arbitrarily fixed to a constant respectively, since it does not affect the values of the barycentric coordinates.

3. Algorithmic realization and a numerical example

Box 3.1 is the three algorithmic steps we have to follow if we are going to solve the twin P4P in terms of Möbius barycentric coordinates.

Box 3.1. Algorithmic steps of solving the twin P4P problem in terms of Möbius barycentric coordinates

"Input data"

(i) observations:

$\{x_l, y_l, -f_l\}$, $\{x_r, y_r, -f_r\}$: left and right image coordinates of the unknown point P to be intersected from the left and right perspective center P_l and P_r.

$\{x_l, y_l, -f_l\}\{x_{l_2}, y_{l_2}, -f_l\}, \{x_{l_3}, y_{l_3}, -f_l\},$

$\{x_{l_4}, y_{l_4}, -f_l\}; \{x_{r_1}, y_{r_1}, -f_r\}, \{x_{r_2}, y_{r_2}, -f_r\},$

$\{x_{r_3}, y_{r_3}, -f_r\}, \{x_{r_4}, y_{r_4}, -f_r\}$:

left and right image coordinates of the four basis points $\{P_1, P_2, P_3, P_4\}$ which generate the double resection.

(ii) *object coordinates* of 4 known points

(*Box 3.1. Continued*)

Step one

"double resection"

(i) computation of the two perspective centers by double resection
(ii) *distance ratios* (2.8)

$$\sqrt{(X_1 - X_l)^2 + (Y_1 - Y_l)^2 + (Z_1 - Z_l)^2}, \cdots,$$

$$\sqrt{(X_4 - X_r)^2 + (Y_4 - Y_r)^2 + (Z_4 - Z_r)^2}$$
from the first set of double resection.

$$s_{l_1} = \frac{\sqrt{(X_1 - X_l)^2 + (Y_1 - Y_l)^2 + (Z_1 - Z_l)^2}}{\sqrt{x_{l_1}^2 + y_{l_1}^2 + f_{l_1}^2}}, \cdots$$

$$s_{r_4} = \frac{\sqrt{(X_4 - X_r)^2 + (Y_4 - Y_r)^2 + (Z_4 - Z_r)^2}}{\sqrt{x_{r_4}^2 + y_{r_4}^2 + f_{r_4}^2}}$$

Step two

"Computation of distance ratios s_l, s_r"

(i) *area elements*

$$area_l\{p_2, p_3, p_4\} = \frac{1}{2}\begin{vmatrix} x_{l_2} & x_{l_3} & x_{l_4} \\ y_{l_2} & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_{l_2} - x_{l_3} & x_{l_2} - x_{l_4} \\ y_{l_2} - y_{l_3} & y_{l_2} - y_{l_4} \end{vmatrix}$$

$$area_l\{p, p_3, p_4\} = \frac{1}{2}\begin{vmatrix} x_l & x_{l_3} & x_{l_4} \\ y_l & y_{l_3} & y_{l_4} \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_l - x_{l_3} & x_l - -x_{l_4} \\ y_l - y_{l_3} & y_l - y_{l_4} \end{vmatrix}$$

Continued

Table. 1. Photographic parameters

Flight height:	ca. 2250m
Focal length:	88.94mm
Frame size:	230mm*230mm
Camera:	RC-10
Overlap:	ca. 65%
Known points:	4, one at each corner

Table 2. Residuals relative to the best values (meters) by least squares adjustment

Point	Barycentric algorithm			Collinear algorithm		
No.	dX	dY	dZ	dX	dY	dZ
1	−1.70	1.60	−0.11	−1.60	1.00	0.50
2	−1.82	1.38	1.82	−1.90	1.52	1.37
3	−1.94	1.17	−1.03	−2.19	1.25	1.51
4	−1.09	−0.36	−3.12	−2.67	1.22	2.25
5	−2.43	1.44	0.87	−2.44	1.59	1.88
RMS	1.85	1.27	1.72	2.19	1.33	1.61

(*Box 3.1. Continued*)

(ii) *computation of matrix* $\boldsymbol{A} \in \mathscr{R}^{2\times 2}$
(2.20) – (2.23)
(iii) *computation of vector* $\mathbf{y} \in \mathscr{R}^{2\times 1}$
(2.18) – (2.19)
(iv) *computation of the inverse equation*

$x = \boldsymbol{A}^{-1}\mathbf{y} : \{s_l, s_r\}$

"Step three"

"coordinate computation"

(i) *computation of Möbius barycentric coordinates* Λ_1 (2.24), Λ_2 (2.25), Λ_3 (2.26), Λ_4 (2.27)
(ii) *computation of Cartesian coordinates*: $\{X, Y, Z\}$ (1.5)

"Output"

"output data"

unknown placement vector: $\{X, Y, Z\}$

It has to be emphasized that the basic *area elements* are invariant with respect to a *translation and a rotation* of the image coordinates. In addition, any factor on s_l, s_r - say cs_l, cs_r - does not change the computational results of *Möbius barycentric coordinates* of the intersected point P.

In summary, this result reveals the beneficial fact that the *Möbius barycentric coordinates* $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ can be equivalently calculated under an arbitrary scale. Moreover, the solution of the twin P4P problem in terms of *Möbius barycentric coordinates* does *not* need the *complete* solution of the three dimensional resection problem for the left and right perspective center P_l and P_r : *Only the first step* of the three dimensional resection problem, namely the *computation of distances* from the unknown resected point to known basis points, has to be implemented. This accounts for a solution of the Grunert equations (E. Grafarend, P. Lohse and B. Schaffrin, 1989). *Step two and three* of the solution of the three dimensional resection problem, namely, the *three dimensional coordinates* of P_l and P_r, and *orientation parameters*, are *not* necessary any more.

Finally, to illustrate our *twin P4P* algorithm, we present a numerical example with an aerial stereopair, whose photographic parameters are listed in Tab.1. The stereo-pair is observed with an analytical plotter. The corresponding 3D object coordinates obtained during this observation are treated as "best values" with which our algorithm is compared. In order to testify the compatibility of our algorithm with the traditional collinear equation algorithm (bundle solution), residuals relative to the "best values" of both algorithms are included in Tab .2, which clearly shows a tolerable coherence of our algorithm with the traditional one, while our algorithm makes unnecessary the orientation parameters and perspective centers.

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