On the Local Times and Boundary Properties of Reflected Diffusions with Jumps in the Positive Orthant

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Abstract

In this paper we study the boundary characteristics of reflected diffusions with jumps in the positive orthant. We consider a model with oblique reflections and study the reflection map in terms of the local time at the boundary (or reflection) faces of $\mathbb{R}^n_+$. In particular, we show that these last processes coincide (up to a multiplicative constant) and that they do not charge the set of times spent by the reflected diffusion in the intersection of two or more boundary faces, so generalizing some of the results that have been shown for semi-martingale reflecting Brownian motions (SRBMs) on the orthant. Moreover, we show that the probability distribution of the reflected diffusion at time t is null over any region contained in the boundary faces, for Lebesgue-a.e. t in $\mathbb{R}_+$, under mild conditions on the diffusion matrix. The paper concludes with extending the results to hyper-rectangles in the positive orthant.

Key words: Diffusions, jumps, reflection maps, local time, semi-martingales
PACS: 60J50, 60J55, 60J60, 60J75, 90B22

1 Introduction

Reflected diffusions with jumps arise in a wide variety of applications such as finance, queueing and risk theory, and models of manufacturing plants. For example, Chen and Whitt (1993) have shown that, in heavy traffic, the process of the number of customers in an open queueing network subject to service interruptions can be approximated by a reflected Brownian motion with jumps in the positive orthant. More recently, such processes have also been shown to arise

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in diffusion limits involving jumps with heavy tailed distributions, Whitt (2002).

Reflected diffusion models with jumps are natural generalizations of the class of so-called piecewise deterministic Markov processes, Davis (1984). The generalization is that the diffusive component adds to the randomness of the evolution of a process between jumps and reflections guarantee that the components of the process stays within a given region as, for example, in queueing networks where the processes are non-negative. These models are also of interest in the risk and insurance context where the jumps could be the claims while the diffusion arises due to volatility of the interest rates, etc. They also play an important role in mathematical finance in the context of barrier options.

A special case of reflected diffusions, namely semi-martingale reflecting Brownian motion (SRBM), has been studied quite extensively due to its importance in models of queueing networks in heavy traffic, Harrison and Williams (1987); Williams (1995, 1998). In Reiman and Williams (1988), the authors gave necessary conditions for the existence of SRBM on the orthant in terms of a special condition on the reflection matrix called the completely-S property. In Taylor and Williams (1993), they established the sufficiency and uniqueness under this condition. Moreover, in Reiman and Williams (1988), they also showed a boundary property in that the regulator process (or reflection map) does not charge the set of times spent by SRBM in the intersection of two or more faces. More recently, in Shen et al. (2002), the authors use this property to develop numerical methods for computing the stationary distribution of queueing networks in heavy traffic.

In Mazumdar and Guillemin (1996), one-dimensional reflected diffusions with jumps are considered. In that case it is shown that not only is the Lebesgue measure of the times of the process at the origin equal to zero, but also that there is no probability mass at that point, Lebesgue-a.e. in t. Moreover, the reflection map is characterized in terms of the local time at level 0. These properties were then used to derive the exact forward equations for reflected diffusions with jumps with exact boundary conditions.

This paper is devoted to the multi-dimensional case. In particular, not only do we prove a generalization of the boundary property in Reiman and Williams (1988) (under an additional invertibility condition on the reflection matrix), but we also obtain the corresponding generalizations of the results shown for the one-dimensional case in Mazumdar and Guillemin (1996).

The organization of this paper is as follows: in Section 2 we introduce the model and obtain the preliminary results. In Section 3 we obtain the main results. In Section 4 we show how our results naturally extend to hyper-rectangles in $\mathbb{R}^n_+$. Finally, Section 5 offers some further comments on the scope of the results.

2 Problem Formulation and Preliminary Results

Let $n > 1$ be a positive integer and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a filtered probability space satisfying the usual hypotheses, Protter (1990), where the following is defined:

- $(W_t) = (W_t^1, \ldots, W_t^n)$, an n-dimensional, $(\mathcal{F}_t)$-adapted, standard Brownian Motion.

- $\Pi(dt, dz)$, an $(\mathcal{F}_t)$-adapted, $\{0,1\}$-valued random measure over $\mathbb{R}_+ \times \mathbb{R}_+^n$, such that $\forall t \in \mathbb{R}_+$, $\int_0^t \int_{\mathbb{R}_+^n} z\Pi(ds, dz) < +\infty$ (componentwise), *a.s.*

Let $b = (b^i)_{i\in\{1,\ldots,n\}}$ and $\sigma = (\sigma^{ij})_{(i,j)\in\{1,\ldots,n\}^2}$ be Borel measurable mappings from $\mathbb{R}_+ \times \mathbb{R}_+^n$ into $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, respectively. We set $a \equiv \sigma\sigma*$, where $\sigma*$ corresponds to the transpose of the matrix σ. Furthermore, let R be an $n \times n$ P-matrix [1] . Recall a square matrix with real coefficients is said to be a P-matrix if every principal minor is strictly positive. Hence, in particular R has strictly positive diagonal elements and all its principal submatrices are non-singular, i.e., $\forall K \subseteq \{1, \ldots, n\}$, $R^{(K)}$ is invertible, where $R^{(K)}$ denotes the principal submatrix obtained from R by deleting rows and columns with indexes in K. Note that these conditions are satisfied, for example, by any real triangular matrix R with strictly positive diagonal elements or, more generally, by positive definite matrices.

We consider the following problem of reflection:

$$dX_t = b(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + \int_{\mathbb{R}_+^n} z\Pi(dt, dz) + RdZ_t \tag{2.1}$$

where [2] :

- $(X_t) = (X_t^1, \ldots, X_t^n)$ is an $(\mathcal{F}_t)$-adapted, *càdlàg*, $\mathbb{R}_+^n$-valued semi-martingale.

- $(Z_t) = (Z_t^1, \ldots, Z_t^n)$ is an $(\mathcal{F}_t)$-adapted, continuous, $\mathbb{R}_+^n$-valued process, such that $\forall i \in \{1, \ldots, n\}$, (Z_t^i) is non-decreasing, null at zero and $\int_{\mathbb{R}_+} X_s^i dZ_s^i = 0$.

By writing $X_t^i = U_t^i + R^{ii}Z_t^i$ from equation (2.1), the regulator (or reflection map) process Z_t^i of X_t^i at level 0 is given by, Harrison (1985):

$$Z_t^i = \frac{1}{R^{ii}} \sup_{s\in[0,t]} \max\{-U_s^i, 0\}$$

Note that, even though the above characterization is given in Harrison (1985) for continuous processes, it can be extended, of course, to *càdlàg* ones. Furthermore, since the jumps are positive, (Z_t) is still continuous.

[1] P-matrices are completely-S (in the terminology of Reiman and Williams (1988)) and, in addition, their principal submatrices are invertible. This invertibility condition guarantees that the linear complementarity problem, associated with defining the reflection map, has a unique solution, see Berman and Plemmons (1994)

[2] Even though we write elements in $\mathbb{R}_+^n$ as row vectors (for simplicity in the notation), we consider them as columns vectors in all the equations they appear

We assume hereafter that b, σ, Π and R are such that equation (2.1) has a unique strong solution. In particular, σ and b satisfy the usual local Lipschitz and linear growth conditions, Jacod and Shiryaev (1987).

The following additional notation will be used throughout the paper. We write, for any function $f : \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}$, set $K \subseteq \{1, \ldots, n\}$ and vector $u \in \mathbb{R}_+^n$, $f(t, u_K)$ to indicate that the spatial coordinates in f with indexes in K have been set to the respective ones in the vector u. If $r \in \mathbb{R}_+$ (i.e., just a scalar), then by $f(t, r_K)$ we mean that the spatial coordinates in f with indexes in K have all been set to r. In the same way, for u and v in $\mathbb{R}_+^n$, and sets $K, \widetilde{K} \subseteq \{1, \ldots, n\}$, $K \cap \widetilde{K} = \emptyset$, we write $f(t, u_K, v_{\widetilde{K}})$ when the spatial coordinates in f with indexes in K and $\widetilde{K}$ have been set to the ones in u and v, respectively, with the same meaning as before in the case that u or v are just scalars. In all the previous notation the remaining spatial coordinates in f are kept free, and so when we write $f(t, u_K) > 0$ for example, we mean that f at time t is strictly positive for all values of its remaining free spatial coordinates, i.e., $\forall x^i \in \mathbb{R}_+$ and $i \in K^c$, where K^c denotes the complement of K with respect to the index set $\{1, \ldots, n\}$. Moreover, all the previous notation holds for any function $f : \mathbb{R}_+^n \to \mathbb{R}$, i.e., for any function independent of t. Also, for $i \in \{1, \ldots, n\}$, we write $dx^{\neq i}$ to denote $dx^1 \cdots dx^n$ when the $i-th$ differential dx^i is omitted, and $x^{r_{\{i\}}}$ to denote the vector $x \in \mathbb{R}_+^n$ when its $i-th$ coordinate x^i has been set to $r \in \mathbb{R}_+$. As usual, whenever we write a.e. (almost everywhere), without specifying the measure, we mean a.e. with respect to Lebesgue measure in the corresponding real space (clear from the context). In the same way, whenever we write a.s. (almost surely), we mean a.s. with respect to P. Furthermore, $F_t : \mathbb{R}_+^n \to [0, 1]$ denotes the probability distribution function of X at time t. We assume that $F_t(dx)$ admits a density (in the sense of distributions), $\forall t \in \mathbb{R}_+$, denoted as $p_t(x)$, $x \in \mathbb{R}_+^n$. In addition, $([X^i, X^i]_t^c)$ denotes the path by path continuous part of the quadratic variation process $([X^i, X^i]_t)$, and $(L^i(t, r))$ the jointly continuous in t and right continuous in r version of the local time at level $r \in \mathbb{R}_+$ for semi-martingale (X_t^i), $i \in \{1, \ldots, n\}$. Note that, since $\forall t \in \mathbb{R}_+$, $\int_0^t \int_{\mathbb{R}_+^n} z \Pi(ds, dz) < +\infty$ (componentwise), *a.s.*, this version of the local time exists (Protter, 1990, Thm. 56, p.176). Furthermore, by (Protter, 1990, Corollary 3, p. 178), we have $L^i(t, r) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}\{r \leq X_s^i \leq r + \varepsilon\} d[X^i, X^i]_s^c$, *a.s.* Finally, as usual, $\mathbf{1}\{\cdot\}$ denotes the indicator function of the corresponding event in parenthesis and $|K|$ the number of elements in set K.

We now establish some key lemmas for the proofs of the main results to be given in the next section of the paper.

Lemma 2.1 *Fix $t \in \mathbb{R}_+$, and let $i \in K \subseteq \{1, \ldots, n\}$. Assume that $a^{ii}(s, 0_K) > 0$ for a.e. $s \in [0, t]$. Then, we have a.s.:*

$$\lambda\{s \in [0, t] : X_s^k = 0, \forall k \in K\} = 0$$

where $\lambda\{\cdot\}$ denotes Lebesgue measure in $\mathbb{R}_+$, and for a.e. $s \in [0, t]$:

$$F_s(0_K) = 0$$

Proof. (X_s) is *càdlàg* and therefore it has at most a countable number of jumps in any compact subinterval of $\mathbb{R}_+$. Hence, since Lebesgue measure is diffuse, we have *a.s.*:

$$\int_0^t a^{ii}(s,X_s)\mathbf{1}\{X_s^k=0,\forall k\in K\}ds=\int_0^t a^{ii}(s,X_{s-})\mathbf{1}\{X_{s-}^k=0,\forall k\in K\}ds$$

Furthermore, *a.s.*:

$$0\le\int_0^t a^{ii}(s,X_{s-})\mathbf{1}\{X_{s-}^k=0,\forall k\in K\}ds\le\int_0^t a^{ii}(s,X_{s-})\mathbf{1}\{X_{s-}^i=0\}ds$$

But, the integral in the r.h.s. of the above expression equals:

$$\int_0^t \mathbf{1}\{X_{s-}^i=0\}d[X^i,X^i]_s^c$$

and, by (Protter, 1990, Corollary 1, p. 168), this last integral equals, *a.s.*:

$$\int_{-\infty}^{+\infty} L^i(t,r)\mathbf{1}\{r=0\}dr=0$$

from where we conclude the first part of the lemma. As for the second part, we note that by Tonelli's theorem:

$$\mathbb{E}(\int_0^t a^{ii}(s,X_s)\mathbf{1}\{X_s^k=0,\forall k\in K\}ds)=\int_0^t \mathbb{E}(a^{ii}(s,X_s)\mathbf{1}\{X_s^k=0,\forall k\in K\})ds$$

and therefore, for *a.e.* $s\in[0,t]$:

$$\mathbb{E}(a^{ii}(s,X_s)\mathbf{1}\{X_s^k=0,\forall k\in K\})=0$$

from where the second part of the lemma now follows. □

Remark 2.1 *Under the assumptions of Lemma 2.1, if $F_s(\cdot)$ is continuous in s for $s\in[0,t]$, then $F_s(0_K)=0$, $\forall s\in[0,t]$.*

Remark 2.2 *Under the assumptions of Lemma 2.1, for a.e. $s\in[0,t]$ $p_s(\cdot)$ does not contain Dirac's delta (or impulse) functions that put probability mass in the $k-th$ face of $\mathbb{R}_+^n$, $\forall k\in K$, i.e., for a.e. $s\in[0,t]$ there is no probability mass in $\bigcup_{k\in K}\{x\in\mathbb{R}_+^n : x^k=0\}$.*

Lemma 2.2 *Fix $t\in\mathbb{R}_+$, and let $r\in\mathbb{R}_+$, $i\in\{1,\dots,n\}$ and ψ be a Borel measurable map from $\mathbb{R}_+^n$ into $\mathbb{R}$, such that there exists $\eta>0$ for which:*

$$\int_0^t\int_{\mathbb{R}_+^{n-1}}|\psi(r_{\{i\}})|\sup_{r<x^i\le r+\eta}\{a^{ii}(s,x)p_s(x)\}dx^{\neq i}ds<+\infty$$

Furthermore, assume that for a.e. $s\in[0,t]$ and for a.e. $x\in$ support$\{\psi(r_{\{i\}})\}$, $\lim_{x^i\downarrow r}p_s(x)$

exists. Then, we have:

$$\mathbb{E}(\int_0^t \psi(X_s)dL^i(s,r)) = \int_0^t \int_{\mathbb{R}_+^{n-1}} \psi(r_{\{i\}})a^{ii}(s,r_{\{i\}}) \lim_{x^i\downarrow r} p_s(x)dx^{\neq i}ds$$

Proof. By (Protter, 1990, Corollary 1, p. 168), $\forall\varepsilon > 0$ we have *a.s.*:

$$\int_r^{r+\varepsilon} L^i(t,u)du = \int_0^t \mathbf{1}\{r \leq X_{s-}^i \leq r+\varepsilon\}a^{ii}(s,X_{s-})ds$$

But, from the same corollary we have $\int_0^t \mathbf{1}\{X_{s-}^i = r\}a^{ii}(s,X_{s-})ds = 0$, *a.s.* Hence, since Lebesgue measure is diffuse and (X_s) is *càdlàg*, the r.h.s. in this last expression equals:

$$\int_0^t \mathbf{1}\{r < X_s^i \leq r+\varepsilon\}a^{ii}(s,X_s)ds$$

Then, by using Fubini's theorem, for ε sufficiently small ($0 < \varepsilon < \eta$) we have:

$$\int_r^{r+\varepsilon} \mathbb{E}(\int_0^t \psi(X_s^{r_{\{i\}}})dL^i(s,u))du = \int_0^t \mathbb{E}(\psi(X_s^{r_{\{i\}}})\mathbf{1}\{r < X_s^i \leq r+\varepsilon\}a^{ii}(s,X_s))ds$$

But, using again Fubini's theorem, this last integral equals:

$$\int_r^{r+\varepsilon}\int_0^t \int_{\mathbb{R}_+^{n-1}} \psi(r_{\{i\}})a^{ii}(s,x)p_s(x)dx^{\neq i}dsdx^i$$

Therefore, we have:

$$\begin{aligned}
\mathbb{E}(\int_0^t \psi(X_s)dL^i(s,r)) &= \mathbb{E}(\int_0^t \psi(X_s^{r_{\{i\}}})dL^i(s,r)) \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_r^{r+\varepsilon} \mathbb{E}(\int_0^t \psi(X_s^{r_{\{i\}}})dL^i(s,u))du \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_r^{r+\varepsilon} (\int_0^t \int_{\mathbb{R}_+^{n-1}} \psi(r_{\{i\}})a^{ii}(s,x)p_s(x)dx^{\neq i}ds)dx^i
\end{aligned}$$

from where the lemma now follows. □

Lemma 2.3 *Fix $t \in \mathbb{R}_+$, and let $i \in K \subseteq \{1,\ldots,n\}$. Assume that $\exists j \in K$, $j \neq i$, such that $a^{jj}(s,0_{K\setminus\{i\}}) > 0$ for a.e. $s \in [0,t]$. Then, we have a.s.:*

$$\int_0^t \mathbf{1}\{X_{s-}^k = 0, \forall k \in K\}dL^i(s,0) = 0$$

Proof. Since $L^i(\cdot,0)$ is *a.s.* continuous, the random measure it induces in $\mathbb{R}_+$ is *a.s.* diffuse. Therefore, again by the fact that (X_s) is *càdlàg*, we have *a.s.*:

$$\int_0^t \mathbf{1}\{X_{s-}^k = 0, \forall k \in K\}dL^i(s,0) = \int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\}dL^i(s,0)$$

Thus, we have:

$$\mathbb{E}(\int_0^t \mathbf{1}\{X_{s-}^k = 0, \forall k \in K\} dL^i(s,0)) = \mathbb{E}(\int_0^t \mathbf{1}\{X_s^k = 0, \forall k \in K\} dL^i(s,0))$$

By Lemmas 2.1 (see also Remark 2.2) and 2.2, the lemma follows. □

Remark 2.3 *Lemmas 2.1 to 2.3 above hold for general reflection matrix R, as long as equation (2.1) admits solution. The special structure imposed on R will be needed for Lemmas 2.4 and 2.5 below, as well as for the results in the next Section 3 of the paper.*

Lemma 2.4 *$\forall t \in \mathbb{R}_+$, define the $n \times n$ matrix R_t as $R_t^{ij} = R^{ij} \mathbf{1}\{X_{t-}^i = 0\}$ if $j \neq i$, and R^{ii} if not. Then, R_t is non-singular, $\forall t \in \mathbb{R}_+$.*

Proof. Let $t \in \mathbb{R}_+$ and $\Lambda_t = \{i \in \{1, \ldots, n\} : X_{t-}^i \neq 0\}$. Then, if $|\Lambda_t| = n$:

$$det R_t = \prod_{i=1}^n R^{ii}$$

Otherwise:

$$det R_t = \prod_{i \in \Lambda_t} R^{ii} \; det R^{(\Lambda_t)}$$

where, by convention, the product over an empty set is taken to be 1. Therefore, by the assumptions on R we have $det R_t \neq 0$, $\forall t \in \mathbb{R}_+$, and the lemma is proved. □

Lemma 2.5 *There exists a predictable process (r_t), with values on the collection of $n \times n$ real matrices, such that $\forall i \in \{1, \ldots, n\}$ and $\forall t \in \mathbb{R}_+$, we have a.s.:*

$$Z_t^i = \sum_{j=1}^n \int_0^t r_s^{ij} \mathbf{1}\{X_{s-}^j = 0\}(\frac{1}{2} dL^j(s,0) - b^j(s, X_{s-})ds)$$

Moreover, $r_t = R_t^{-1}$, $\forall t \in \mathbb{R}_+$, where R_t is the same matrix as in Lemma 2.4.

Proof. By using equation (2.1) and Meyer-Itô's formula (Protter, 1990, Thm. 51, p.167) with convex function $f(x) = x^+ = max\{0, x\}$, $x \in \mathbb{R}$, we obtain:

$$\frac{1}{2} dL^i(t,0) - \mathbf{1}\{X_{t-}^i = 0\} b^i(t, X_{t-}) = \sum_{j=1}^n \mathbf{1}\{X_{t-}^i = 0\} R^{ij} dZ_t^j$$

Note that the continuous local martingale term resulting from applying Meyer-Itô's formula is identically zero, since it is null at $t = 0$ and has null quadratic variation. Now, since (X_s) is *càdlàg* and $L^i(ds, 0)$ is diffuse, we have *a.s.*:

$$\int_0^t \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0) = \int_0^t \mathbf{1}\{X_s^i = 0\} dL^i(s,0) = L^i(t,0)$$

The lemma now follows by using Lemma 2.4. □

3 Main Results

Using the lemmas of the previous section, we will prove the following two theorems related to the regulator process (Z_t) of (X_t) at level 0.

Theorem 3.1 *Fix $t \in \mathbb{R}_+$, and assume that $\exists i, j \in K \subseteq \{1, \ldots, n\}$, $i \neq j$, such that $a^{ii}(s, 0_{K\setminus\{j\}}) > 0$ and $a^{jj}(s, 0_{K\setminus\{i\}}) > 0$ for a.e. $s \in [0, t]$. Then, $\forall q \in K$ we have a.s.:*

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} dZ^q_s = 0$$

Proof. From Lemma 2.5 we have *a.s.*:

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} dZ^q_s = \sum_{l=1}^n \int_0^t r^{ql}_s \mathbf{1}\{X^k_{s-} = 0, \forall k \in \{l\} \cup K\}(\frac{1}{2} dL^l(s, 0) - b^l(s, X_{s-})ds)$$

Since (X_s) is *càdlàg* and Lebesgue measure is diffuse, we have *a.s.*:

$$\int_0^t r^{ql}_s \mathbf{1}\{X^k_{s-} = 0, \forall k \in \{l\} \cup K\} b^l(s, X_{s-})ds = \int_0^t r^{ql}_s \mathbf{1}\{X^k_s = 0, \forall k \in \{l\} \cup K\} b^l(s, X_{s-})ds$$

Hence, by using Lemma 2.1 we conclude that $\forall l \in \{1, \ldots, n\}$, *a.s.*:

$$\int_0^t r^{ql}_s \mathbf{1}\{X^k_{s-} = 0, \forall k \in \{l\} \cup K\} b^l(s, X_{s-})ds = 0$$

Furthermore, from Lemma 2.3 we conclude that $\forall l \in \{1, \ldots, n\}$, *a.s.*:

$$\mathbf{1}\{X^k_{s-} = 0, \forall k \in \{l\} \cup K\} = 0$$

for $L^l(\cdot, 0) - a.e.$ $s \in [0, t]$. The theorem now follows. □

Remark 3.1 *Theorem 3.1 is proved for the special case of SRBM in (Reiman and Williams, 1988, Theorem 1). We note that this follows trivially from Theorem 3.1 above since, in the SRBM case, $a^{ii} > 0$ (constant) for each $i \in \{1, \ldots, n\}$. Our proof for the general case, Theorem 3.1, is more direct than the one given in Reiman and Williams (1988). However, the assumption that R is a P-matrix is stronger than the completely-S condition used in Reiman and Williams (1988).*

Remark 3.2 *In the same way as in the proof of Theorem 3.1 above, if for some $i \in \{1, \ldots, n\}$, $a^{ii}(s, 0_{\{i\}}) > 0$ for a.e. $s \in [0, t]$, then we have a.s.:*

$$Z^i_t = \frac{1}{2} \int_0^t r^{ii}_s \mathbf{1}\{X^i_{s-} = 0\} dL^i(s, 0)$$

From this last observation, we have the following result.

Theorem 3.2 *Assume that* $\forall i \in \{1,\ldots,n\}$, $a^{ii}(t, 0_{\{i\}}) > 0$ *for a.e.* $t \in \mathbb{R}_+$. *Then,* $\forall i \in \{1,\ldots,n\}$ *we have a.s.:*

$$Z_t^i = \frac{1}{2R^{ii}} L^i(t,0) \quad \forall t \in \mathbb{R}_+$$

i.e., these two processes are indistinguishable.

Proof. Under the above assumptions, from Lemma 2.3 we have $\forall t \in \mathbb{R}_+$ and $\forall i \in \{1\ldots,n\}$, *a.s.*:

$$\mathbf{1}\{X_{s-}^l = 0, for\ some\ l \neq i\} = 0$$

for $L^i(\cdot,0) - a.e.\ s \in [0,t]$. Thus, we have *a.s.*:

$$\int_0^t r_s^{ii} \mathbf{1}\{X_{s-}^l = 0, for\ some\ l \neq i\} \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0) = 0$$

and therefore:

$$\int_0^t r_s^{ii} \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0) = \int_0^t r_s^{ii} \mathbf{1}\{X_{s-}^l > 0, \forall l \neq i\} \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0)$$

But, it is easy to see that on the event $\{X_{s-}^i = 0, X_{s-}^l > 0, \forall l \neq i\}$, $r_s^{ii} = \frac{1}{R^{ii}}$. Whence, from Remark 3.2 we conclude that $\forall t \in \mathbb{R}_+$, *a.s.*:

$$Z_t^i = \frac{1}{2R^{ii}} \int_0^t \mathbf{1}\{X_{s-}^l > 0, \forall l \neq i\} \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0)$$

But, again from Lemma 2.3, *a.s.*:

$$\int_0^t \mathbf{1}\{X_{s-}^l > 0, \forall l \neq i\} \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0) = \int_0^t \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0)$$

and, since (X_s) is *càdlàg* and $L^i(ds,0)$ is diffuse, we have *a.s.*:

$$\int_0^t \mathbf{1}\{X_{s-}^i = 0\} dL^i(s,0) = \int_0^t \mathbf{1}\{X_s^i = 0\} dL^i(s,0) = L^i(t,0)$$

Thus, $\forall t \in \mathbb{R}_+$ we have:

$$Z_t^i = \frac{1}{2R^{ii}} L^i(t,0) \quad a.s.$$

i.e., the above processes are modifications of each other. Indistinguishability now follows from almost surely sample path continuity. □

Corollary 3.1 *Under the assumptions of Theorem 3.2 above,* $\forall i \in \{1,\ldots,n\}$ *we have a.s.:*

$$support\{L^i(\cdot,0)\} = support\{Z_\cdot^i\} \subseteq \{s \in \mathbb{R}_+ :\ X_s^i = 0, X_s^l > 0, \forall l \neq i\}$$

Proof. This follows in a straightforward manner from Lemma 2.3 and Theorem 3.1. □

4 Some Generalizations

The results obtained in the previous sections can readily be generalized to hyper-rectangles in the positive orthant. In that direction, let $c = (c^1, \ldots, c^n)$ with $c^i > 0$, $\forall i \in \{1, \ldots, n\}$. We now consider the following problem of reflection:

$$dX_t = b(t, X_{t-})dt + \sigma(t, X_{t-})dW_t + \int_{\mathbb{R}^n_+} z\Pi(dt, dz) + RdZ_t - \widetilde{R}d\widetilde{Z}_t \tag{4.2}$$

where b, σ, W, Π and Z are as before, and:

- $(X_t) = (X^1_t, \ldots, X^n_t)$ is an $(\mathcal{F}_t)$-adapted, *càdlàg*, $\times_{i=1}^n[0, c^i]$-valued semi-martingale.
- $(\widetilde{Z}_t) = (\widetilde{Z}^1_t, \ldots, \widetilde{Z}^n_t)$ is an $(\mathcal{F}_t)$-adapted, continuous, $\mathbb{R}^n_+$-valued process, such that $\forall i \in \{1, \ldots, n\}$, $(\widetilde{Z}^i_t)$ is non-decreasing, null at zero and $\int_{\mathbb{R}_+}(c^i - X^i_s)d\widetilde{Z}^i_s = 0$.

Furthermore, R and $\widetilde{R}$ are $n \times n$ P-matrices satisfying the following additional condition. Let M be the matrix with blocks $M^{11} = R$, $M^{12} = -\widetilde{R}$, $M^{21} = -R$ and $M^{22} = \widetilde{R}$. Then, we assume that any principal submatrix of M with dimension of at most $n \times n$ is non-singular, i.e., $\forall K \subseteq \{1, \ldots, 2n\}$, $|K| \geq n$, $M^{(K)}$ is invertible, where $M^{(K)}$ denotes, as before, the principal submatrix obtained from M by deleting rows and columns with indexes in K. Note that these conditions are satisfied, for example, by real triangular matrices R and $\widetilde{R}$ with strictly positive diagonal elements.

In addition, if the jump introduced by Π at time t is such that $X^i_t > c^i$, then we assume that X^i_t is set to c^i at that instant. This assumption is motivated from a queueing theory point of view when work arrives to a finite queue. There, if X^i_t represents for example the buffer content in the $i - th$ network element at time t and c^i the maximum buffer allocation allowed in that element, then work that arrives in excess of this maximum is lost, see for example Baccelli and Brémaud (2003). Hence, we may assume hereafter that $\forall t \in \mathbb{R}_+$, $\int_{t-}^t \int_{\mathbb{R}^n_+} z\Pi(ds, dz) \leq (c - X_{t-})$, *a.s.*, where this last inequality is of course understood to hold componentwise.

We note that the model contained in equation (4.2) corresponds to a generalization of the respective one in equation (2.1), in the sense that equation (2.1) can be obtained from equation (4.2) by considering the limiting case when $c^i \to \infty$, $\forall i \in \{1, \ldots, n\}$.

Also, by writing $X^i_t = U^i_t + R^{ii}Z^i_t$ from equation (4.2), the regulator process Z^i_t of X^i_t at level 0 is given by, Harrison (1985):

$$Z^i_t = \frac{1}{R^{ii}} \sup_{s\in[0,t]} \max\{-U^i_s, 0\}$$

and, by writing $X_t^i = V_t^i - \widetilde{R}^{ii}\widetilde{Z}_t^i$ from equation (4.2), the regulator process $\widetilde{Z}_t^i$ of X_t^i at level c^i is given by, Harrison (1985):

$$\widetilde{Z}_t^i = \frac{1}{\widetilde{R}^{ii}} \sup_{s\in[0,t]} \max\{V_s^i - c^i, 0\}$$

Again, note that even though the above characterizations are given in Harrison (1985) for continuous processes, they can be extended, of course, to *càdlàg* ones. Furthermore, since the jumps are positive and $\forall t \in \mathbb{R}_+$, $\int_{t-}^t \int_{\mathbb{R}_+^n} z\Pi(ds, dz) \leq (c - X_{t-})$ (componentwise), *a.s.*, (Z_t) and $(\widetilde{Z}_t)$ are still continuous.

Like in the previous case, we assume hereafter that σ, b, Π, R and $\widetilde{R}$ are such that equation (4.2) has a unique strong solution. In particular, like before, σ and b satisfy the usual local Lipschitz and linear growth conditions, Jacod and Shiryaev (1987).

We now establish the corresponding generalizations of Lemmas 2.1 and 2.3, in Lemmas 4.1 and 4.2 below, respectively.

Lemma 4.1 *Fix $t \in \mathbb{R}_+$, and let $K, \widetilde{K} \subseteq \{1, \ldots, n\}$ be such that $\widetilde{K} \cap K = \emptyset$. Assume that $\exists i \in K \cup \widetilde{K}$ such that $a^{ii}(s, 0_K, c_{\widetilde{K}}) > 0$ for a.e. $s \in [0, t]$. Then, we have a.s.:*

$$\lambda\{s \in [0, t] : X_s^q = 0, \forall q \in K \cup \widetilde{K}\} = 0$$

where $\lambda\{\cdot\}$ denotes Lebesgue measure in $\mathbb{R}_+$, and for a.e. $s \in [0, t]$:

$$F_s(0_K, c_{\widetilde{K}}) = 0$$

Proof. Follows by exactly the same arguments as in the proof of Lemma 2.1. □

Remark 4.1 *Under the assumptions of Lemma 4.1, if $F_s(x)$ is continuous in s for $s \in [0, t]$, then $F_s(0_K, c_{\widetilde{K}}) = 0$, $\forall s \in [0, t]$.*

Remark 4.2 *Under the assumptions of Lemma 4.1, for a.e. $s \in [0, t]$ $p_s(\cdot)$ does not contain Dirac's delta (or impulse) functions that put probability mass in $\bigcup_{k\in K}\{x \in \mathbb{R}_+^n : x^k = 0\}$ or $\bigcup_{q\in\widetilde{K}}\{x \in \mathbb{R}_+^n : x^q = c^q\}$, i.e., for a.e. $s \in [0, t]$ there is no probability mass in those faces.*

Note that, by right continuity of $L^i(t, \cdot)$, $\forall t \in \mathbb{R}_+$ we have $L^i(t, c^i) = 0$, *a.s.* Hence, in the following results we will consider:

$$L^i(t, c^i-) = \lim_{r\uparrow c^i} L^i(t, r) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}\{c^i - \varepsilon \leq X_s^i \leq c^i\} d[X^i, X^i]_s^c$$

where, since $\forall t \in \mathbb{R}_+$, $\int_0^t \int_{\mathbb{R}_+^n} z\Pi(ds, dz) < +\infty$ (componentwise), *a.s.*, the above limits exist and are equal by (Protter, 1990, Theorem 56 and Corollary 3, p. 176 and 178, respectively). Note that $L^i(\cdot, c^i-)$ is still continuous and non-decreasing, *a.s.* (see the proof of the above mentioned

Theorem 56).

Lemma 4.2 *Fix $t \in \mathbb{R}_+$, and let $K, \widetilde{K} \subseteq \{1, \ldots, n\}$ be such that $\widetilde{K} \cap K = \emptyset$. If $i \in K$ and $\exists j \in K \cup \widetilde{K}$, $j \neq i$, such that $a^{jj}(s, 0_{K \setminus \{i\}}, c_{\widetilde{K}}) > 0$ for a.e. $s \in [0, t]$, then we have a.s.:*

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} \mathbf{1}\{X^q_{s-} = c^q, \forall q \in \widetilde{K}\} dL^i(s, 0) = 0$$

In the same way, if $i \in \widetilde{K}$ and $\exists j \in K \cup \widetilde{K}$, $j \neq i$, such that $a^{jj}(s, 0_K, c_{\widetilde{K} \setminus \{i\}}) > 0$ for a.e. $s \in [0, t]$, then we have a.s.:

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} \mathbf{1}\{X^q_{s-} = c^q, \forall q \in \widetilde{K}\} dL^i(s, c^i-) = 0$$

Proof. Follows by exactly the same arguments as in the proof of Lemma 2.3. We note that Lemma 2.2 can trivially be extended to the case where the integrator is $L^i(ds, r-)$, $r \in \mathbb{R}_+$, instead of $L^i(ds, r)$. □

Remark 4.3 *Lemmas 4.1 and 4.2 above hold for general reflection matrices R and $\widetilde{R}$, as long as equation (4.2) admits solution. The special structure imposed on R, $\widetilde{R}$ and M will be needed for Lemma 4.3 below, which corresponds to the generalization of Lemma 2.4, as well as for all the results that follows it. Using this last generalization, we will obtain Theorem 4.1, also below, which in turn corresponds to the respective generalization of Theorem 3.2 in the previous section.*

Lemma 4.3 *$\forall t \in \mathbb{R}_+$, define the $n \times n$ matrices R_t, $\widetilde{R}_t$, ρ_t and $\tilde{\rho}_t$ as: $R^{ij}_t = R^{ij} \mathbf{1}\{X^i_{t-} = 0\}$ if $i \neq j$, R^{ii} if not; $\widetilde{R}^{ij}_t = \widetilde{R}^{ij} \mathbf{1}\{X^i_{t-} = c^i\}$ if $i \neq j$, $\widetilde{R}^{ii}$ if not; $\rho^{ij}_t = -R^{ij} \mathbf{1}\{X^i_{t-} = c^i\}$; and $\tilde{\rho}^{ij}_t = -\widetilde{R}^{ij} \mathbf{1}\{X^i_{t-} = 0\}$. Furthermore, $\forall t \in \mathbb{R}_+$ set M_t as the matrix with blocks $M^{11}_t = R_t$, $M^{12}_t = \tilde{\rho}_t$, $M^{21}_t = \rho_t$ and $M^{22}_t = \widetilde{R}_t$. Then, M_t is non-singular, $\forall t \in \mathbb{R}_+$.*

Proof. Let $t \in \mathbb{R}_+$, $\Lambda_t = \{i \in \{1, \ldots, n\} : X^i_{t-} \neq 0\}$ and $\widetilde{\Lambda}_t = \{i \in \{n+1, \ldots, 2n\} : X^{i-n}_{t-} \neq c^{i-n}\}$. Then, if $|\Lambda_t \cup \widetilde{\Lambda}_t| = 2n$:

$$det M_t = \prod_{i=1}^n R^{ii} \widetilde{R}^{ii}$$

Otherwise:

$$det M_t = \prod_{i \in \Lambda_t} R^{ii} \prod_{\substack{j \text{ such that} \\ j+n \,\in\, \widetilde{\Lambda}_t}} \widetilde{R}^{jj} \; det M^{(\Lambda_t \cup \widetilde{\Lambda}_t)}$$

where, by convention, the product over an empty index set is 1. Therefore, by the assumptions on R, $\widetilde{R}$ and M we have $det M_t \neq 0$, $\forall t \in \mathbb{R}_+$, and the lemma is proved. □

Theorem 4.1 *Assume that $\forall i \in \{1, \ldots, n\}$, $a^{ii}(t, 0_{\{i\}}) > 0$ and $a^{ii}(t, c_{\{i\}}) > 0$ for a.e. $t \in \mathbb{R}_+$. Then, $\forall i \in \{1, \ldots, n\}$ we have a.s.:*

$$Z^i_t = \frac{1}{2R^{ii}} L^i(t, 0) \quad \forall t \in \mathbb{R}_+$$

and:

$$\widetilde{Z}_t^i = \frac{1}{2\widetilde{R}^{ii}} L^i(t, c^i -) \quad \forall t \in \mathbb{R}_+$$

i.e., the corresponding processes above are indistinguishable.

Proof. Since (X_s) is *càdlàg*, Lebesgue measure is diffuse and $\forall t \in \mathbb{R}_+$, $\int_{\mathbb{R}_+^n} z\Pi(t, dz) \leq (c - X_t)$ (componentwise), *a.s.*, by using equation (4.2), Lemma 4.1 and Meyer-Itô's formula (Protter, 1990, Thm. 51, p. 167) with convex function $f(x) = x^+ = max\{0, x\}$, $x \in \mathbb{R}$, we find:

$$\frac{1}{2} dL^i(t, 0) = \sum_{j=1}^n \mathbf{1}\{X_{t-}^i = 0\} R^{ij} dZ_t^j - \sum_{j=1}^n \mathbf{1}\{X_{t-}^i = 0\} \widetilde{R}^{ij} d\widetilde{Z}_t^j \tag{4.3}$$

Note that the continuous local martingale term resulting from applying Meyer-Itô's formula is identically zero, since it is null at $t = 0$ and has null quadratic variation. Now, we observe that semi-martingale X_t^i can be uniquely decomposed as:

$$X_t^i = X_0^i + M_t^i + A_t^i + J_t^i$$

where the continuous local martingale M_t^i, $M_0^i = 0$, the bounded variation process A_t^i, $A_0^i = 0$, and the jump process J_t^i, $J_0^i = 0$, are given by:

$$M_t^i = \sum_{j=1}^n \int_0^t \sigma^{ij}(s, X_{s-}) dW_s^j \qquad A_t^i = \int_0^t b^i(s, X_{s-}) ds + \sum_{j=1}^n R^{ij} Z_t^j - \sum_{j=1}^n R^{ij} \widetilde{Z}_t^j$$

and:

$$J_t^i = \int_0^t \int_{\mathbb{R}_+^n} z^i \Pi(ds, dz)$$

Since $\forall t \in \mathbb{R}_+$, $L^i(t, c^i) = 0$ and $J_t^i < +\infty$, *a.s.*, then, by using (Protter, 1990, Cor. 1, p. 177) we have:

$$L^i(t, c^i -) = -2 \int_0^t \mathbf{1}\{X_{s-}^i = c^i\} dA_s^i$$

from where we obtain, by using Lemma 4.1 and the facts that (X_s) is *càdlàg* and Lebesgue measure is diffuse:

$$\frac{1}{2} dL^i(t, c^i -) = -\sum_{j=1}^n \mathbf{1}\{X_{t-}^i = c^i\} R^{ij} dZ_t^j + \sum_{j=1}^n \mathbf{1}\{X_{t-}^i = c^i\} \widetilde{R}^{ij} d\widetilde{Z}_t^j \tag{4.4}$$

Note that, even though the term $j = i$ in the first sum above is zero (since Π is non-negative), we have retained it for convenience. Now, from Lemma 4.3 we conclude that equations (4.3) and (4.4), $i = 1, \ldots, n$, have a unique solution for every $t \in \mathbb{R}_+$. This corresponds to the generalization of Lemma 2.5. Thus, from Lemma 4.2 we conclude that $\forall t \in \mathbb{R}_+$, *a.s.*:

$$Z_t^i = \frac{1}{2R^{ii}} L^i(t, 0) \qquad \widetilde{Z}_t^i = \frac{1}{2\widetilde{R}^{ii}} L^i(t, c^i -)$$

Indistinguishability now follows from almost surely sample path continuity. □

Corollary 4.1 *Under the assumptions of Theorem 4.1, $\forall i \in \{1,\ldots,n\}$ we have a.s.:*

$$support\{L^i(\cdot,0)\} = support\{Z^i_\cdot\} \subseteq \{s \in \mathbb{R}_+ : \ X^i_s = 0, X^l_s \notin \{0, c^l\}, \forall l \neq i\}$$

and:

$$support\{L^i(\cdot,c^i-)\} = support\{\widetilde{Z}^i_\cdot\} \subseteq \{s \in \mathbb{R}_+ : \ X^i_s = c^i, X^l_s \notin \{0, c^l\}, \forall l \neq i\}$$

Proof. Straightforward from Lemma 4.2 and Theorem 4.1. □

The next corollary gives the generalization of Theorem 3.1 in the previous section.

Corollary 4.2 *Fix $t \in \mathbb{R}_+$, and let $K, \widetilde{K} \subseteq \{1,\ldots,n\}$ be such that $\widetilde{K} \cap K = \emptyset$. Under the assumptions of Theorem 4.1, and if furthermore $i \in K$ and $|\widetilde{K} \cap K| \geq 2$, then we have a.s.:*

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} \mathbf{1}\{X^q_{s-} = c^q, \forall q \in \widetilde{K}\} dZ^i_s = 0$$

In the same way, under the assumptions of Theorem 4.1, and if furthermore $i \in \widetilde{K}$ and $|\widetilde{K} \cap K| \geq 2$, then we have a.s.:

$$\int_0^t \mathbf{1}\{X^k_{s-} = 0, \forall k \in K\} \mathbf{1}\{X^q_{s-} = c^q, \forall q \in \widetilde{K}\} d\widetilde{Z}^i_s = 0$$

Proof. Straightforward from Lemma 4.2 and Theorem 4.1. □

Remark 4.4 *Of course, Theorem 3.1 in the previous section can be generalized for the case where the assumptions of Theorem 4.1 do not hold $\forall i \in \{1,\ldots,n\}$, but only for $i \in K \subset \{1,\ldots,n\}$. For simplicity we have chosen to consider here just the case in Corollary 4.2 above, being the extension of this result straightforward from Lemmas 4.2 and 4.3.*

5 Concluding Remarks

In this paper we have assumed that the jumps are positive. A little reflection should show that the results are readily generalizable to processes with both, positive and negative jumps, with the requirement that $\forall t \in \mathbb{R}_+$, $\int_0^t \int_{\mathbb{R}^n} |z|\Pi(ds,dz) < +\infty$ (componentwise), *a.s.* Jumps that cross the lower boundary at 0 must be truncated as we truncated the jumps that cross a certain level in Section 4. Therefore, in addition, Π can be assumed to be such that, $\forall t \in \mathbb{R}_+$, $\int_{t-}^t \int_{\mathbb{R}^n} z\Pi(ds,dz) \geq -X_{t-}$ (componentwise), *a.s.*, and then the regulator process (Z_t) (of (X_t) at level 0) is still continuous. The results can then be obtained mutatis mutandis. Models with negative jumps are important in risk theory or in financial models with claims arising at random times. Finally, we note that the results exposed in this paper are not only interesting from the point of view of defining the boundary behavior and obtaining a local time characterization of the reflection map, but also they are of use in characterizing the stationary distributions (when they exist) of reflected diffusions with jumps. These and other issues will be presented elsewhere.

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