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ON PRODUCT-FORM STATIONARY DISTRIBUTIONS FOR REFLECTED DIFFUSIONS WITH JUMPS IN THE POSITIVE ORTHANT

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Abstract

In this paper we study the stationary distributions for reflected diffusions with jumps in the positive orthant. Under the assumption that the stationary distribution possesses a density in $\mathbb{R}_+^n$ that satisfies certain finiteness conditions, we characterize the Fokker-Planck equation. We then provide necessary and sufficient conditions for the existence of a product-form distribution for diffusions with oblique boundary reflections and jumps. For this we exploit a recent characterization of the boundary properties of such reflected processes. In particular, we show that the conditions generalize those for semi-martingale reflecting Brownian motions (SRBMs) and reflected Lévy processes. We provide explicit results for some models of interest.

Keywords: Diffusions; jumps; reflection maps; local time; semi-martingales; stationary distributions; product-form

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1. Introduction

Reflected diffusions with jumps arise in a wide variety of applications such as finance, queueing and risk theory, and models of manufacturing plants. For example, in (3) the authors have shown that, in heavy traffic, the process corresponding to the number of customers in an open queueing network subject to service interruptions can be approximated by a reflected Brownian motion with jumps in the positive orthant. More recently, such processes have also been shown to arise in diffusion limits involving jumps with heavy tailed distributions, (15). In addition, reflected diffusion models with jumps are natural generalizations of the class of so-called piecewise deterministic Markov processes, (4). The generalization being that the diffusive component adds to the randomness of the evolution of a process between jumps, and reflections guarantee that the components of the process stay within a given region (for example, in queueing networks where the processes are non-negative). These models are also of interest in the risk and insurance context, where the jumps could be the claims while the diffusion arises due to volatility of the interest rates, etc. They also play an important role in mathematical finance, in the context of barrier options for example.

A special case of reflected diffusions, namely semi-martingale reflecting Brownian motion (SRBM), has been studied quite extensively due to its importance in models of queueing networks in heavy traffic, (5; 16; 17). In (12), necessary conditions for the existence of SRBM on the positive orthant are given, in terms of a special condition on the reflection matrix called the completely-S property. In (14), the sufficiency and uniqueness under this condition are established. Moreover, in (12) it is also shown a boundary property in that the reflection map does not charge the set of times spent by SRBM in the intersection of two or more faces. More recently, in (13) the authors use this property to develop numerical methods for computing the stationary distribution of queueing networks in the heavy traffic limit.

In (9), one-dimensional reflected diffusions with jumps, and their corresponding stationary distributions (when they exist), are studied. It is shown that not only is the Lebesgue measure of the set of times that the process expends at the origin equal to zero, but also that there is no probability mass at that point, Lebesgue-a.e. in t. Recently, in (10) these properties, along with some new ones regarding the reflection map and local times at the boundary, have been extended to the multi-dimensional case. The boundary characterization is essential to study the properties of the stationary distribution of the process, when it exists.

There has been much interest in the particular case when the marginal distributions are independent, termed as product-form distributions. Product-form distributions for stationary, reflected Lévy processes (constant drift and diffusion matrix) have been exploited in (1) for example, however no conditions to have this separable setting were given. In (5) the authors provide necessary and sufficient conditions for the separability of the distribution in stationary regime of SRBM, where there are no jumps and the drift, as well as the diffusion matrix, are constant. Negative results for queueing models with general independent Lévy inputs are given in (7), and more recently in (8).

In this paper we exploit the boundary characterization in (10) to study the stationary distributions of reflected diffusions with jumps in $\mathbb{R}_+^n$. In particular, we derive the forward or Fokker-Planck equation. We then study the conditions for the existence of product-form stationary distributions, assuming the stationary setting exists, obtaining the necessary and sufficient conditions for the case of non-constant drift and diffusion coefficients, and so generalizing the conditions found in (5) for the simpler case of SRBM, and for reflected Lévy processes, (7; 8).

The organization of this paper is as follows: In Section 2 we introduce the model and obtain the preliminary results. In Section 3 we specialize the results for the case where the stationary density is separable. In Section 4 we obtain the main result, regarding necessary and sufficient conditions for product-form distributions of reflected diffusions with jumps in the positive orthant in stationary regime, assuming the stationary setting and its corresponding density exist. In Section 5 we consider some examples of interest. Finally, Section 6 offers some further comments on the scope of the results.

2. Problem Formulation and Preliminary Results

Let $n > 1$ be a positive integer, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a filtered probability space satisfying the usual hypotheses (see (11)), and $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n / x_i \geq 0, \forall i = 1, \ldots, n\}$ (the same corresponding definition applies for $\mathbb{R}_+$, i.e., it includes 0). Note that, even though when explicitly writing elements in $\mathbb{R}^n$ or $\mathbb{R}_+^n$ we write them as row vectors (for simplicity in the notation), they are treated as columns vectors in all the equations they appear. We consider the following problem of reflection in $\mathbb{R}_+^n$:

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dW_t + \int_{\mathbb{R}_+^n} z\Pi(dt, dz) + RdZ_t \tag{2.1}$$

where:

- $(X_t) = (X_t^1, \ldots, X_t^n)$ is an $(\mathcal{F}_t)$-adapted, *càdlàg*, $\mathbb{R}_+^n$-valued semi-martingale.
- $(W_t) = (W_t^1, \ldots, W_t^n)$ is an n-dimensional, $(\mathcal{F}_t)$-adapted, standard Brownian Motion.
- $(Z_t) = (Z_t^1, \ldots, Z_t^n)$ is an $(\mathcal{F}_t)$-adapted, continuous, $\mathbb{R}_+^n$-valued process, such that $\forall i \in \{1, \ldots, n\}$, (Z_t^i) is non-decreasing, null at zero and $\int_{\mathbb{R}_+} X_s^i dZ_s^i = 0$.
- $b = (b_i)_{i \in \{1,\ldots,n\}}$ and $\sigma = (\sigma_{i,j})_{i,j \in \{1,\ldots,n\}}$ are Borel measurable mappings from $\mathbb{R}_+^n$ into $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, respectively. We set $a \triangleq \sigma\sigma*$, where $\sigma*$ corresponds to the transpose of matrix σ. Furthermore, and denoting as $a_{i,i}(0_i)$ the $i-th$ diagonal element of matrix a when its $i-th$ coordinate has been set to zero, we assume that $\forall i \in \{1, \ldots, n\}$, $a_{i,i}(0_i) > 0$, everywhere in $\mathbb{R}_+^{n-1}$.
- $\Pi(dt, dz)$ is an $(\mathcal{F}_t)$-adapted, $\{0, 1\}$-valued random measure over $\mathbb{R}_+ \times \mathbb{R}_+^n$. Furthermore, we assume that $\Pi(dt, dz)$ admits a predictable compensator $\Pi^p(dt, dz)$ of the form $\lambda(\omega)\overline{K}(\omega, t, dz)dt$, such that: 1) $\overline{K}(\omega, t, dz)$ is a Markovian, predictable transition kernel of $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))$, where $\mathcal{P}$ denotes the corresponding predictable σ-field on $\Omega \times \mathbb{R}_+$, taking the form $K(X_{t-}(\omega), dz)$ with $K(x, dz)$ a transition kernel of $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n))$ into itself, such that $\forall x \in \mathbb{R}_+^n$, it admits a density (in the sense of distributions) $k(x, z)$; and, 2) in the stationary setting (assuming it exits) the intensity $\lambda(\omega)$ is jointly distributed with $X_{t-}(\omega)$ as $\psi(dr, dx)$, where the variables $r \in \mathbb{R}_+$ and $x \in \mathbb{R}_+^n$ refer to λ and X_{t-}, respectively, and the joint law $\psi(dr, dx)$ admits a density (in the sense of distributions) $\varphi(r, x)$. Moreover, $\forall x \in \mathbb{R}_+^n$, we write $\Phi(x) \triangleq \int_{\mathbb{R}_+} r\varphi(r, x)dr$.
- R is an $n \times n$ P-matrix. Recall a square matrix with real coefficients is said to be a P-matrix if every principal minor is strictly positive (note that this condition is satisfied, for example, by any real triangular matrix R with strictly positive diagonal elements or, more generally, by positive definite matrices). Hence, in

> particular R has strictly positive diagonal elements and it is invertible. Also, note that P-matrices are completely-S (in the terminology of (12)) but, in addition, their principal submatrices are invertible. This invertibility condition guarantees that the linear complementarity problem, associated with defining the reflection map, has a unique solution (see (2)).

In addition, we assume hereafter that b, σ, Π and R are such that equation (2.1) has a unique strong solution. In particular, b and σ satisfy the usual local Lipschitz and linear growth conditions, (6).

The following notation will be used throughout the paper. Let $\mathbb{N}$ be the set of strictly positive integers. Then, for $k \in \mathbb{N}$, $f : \mathbb{R}_+^k \to \mathbb{R}$ and $i \in \{1, \ldots, k\}$, we write $f(0_i)$ to indicate that the $i-th$ coordinate in f has been set to zero and $f(0_i^+)$, or $\mathcal{L}_k[f](\cdot)$, to denote $\lim_{x_i \downarrow 0} f(x)$, or the (k-dimensional) Laplace transform of function f, respectively. Furthermore, whenever we write a.e. (almost everywhere) in $\mathbb{R}_+^k$ or for a.e. (almost every) $x \in \mathbb{R}_+^k$, without specifying the measure, we mean a.e. with respect to Lebesgue measure in $\mathbb{R}_+^k$. Of course, when $k = 1$, the indices i and k in all the previous notation are not necessary and will be omitted. Moreover, for $i, j \in \{1, \ldots, n\}$, $([X^i, X^j]_t^c)$ denotes the path by path continuous part of the quadratic covariation process $([X^i, X^j]_t)$, or of the quadratic variation process if $i = j$, and (L_t^i) the local time at level 0 for semi-martingale (X_t^i). In addition, for l and $k \in \mathbb{N}$, we write $\mathcal{C}^l(\mathcal{D}^k)$ (resp., $\mathcal{C}_b^l(\mathcal{D}^k)$) to indicate the set of all functions $f : \mathbb{R}_+^k \to \mathbb{R}$ which, along with all their partial derivatives up to order l, inclusive, are continuous (resp., bounded and continuous) in $\mathcal{D}^k \subseteq \mathbb{R}_+^k$. Of course, when no superindex l is specified in this last notation we refer to f itself, excluding its derivatives. Also, to simplify the expressions, for $k \in \{1, \ldots, n\}$ we write $dx_{\neq k}$ to indicate $dx_1 \cdots dx_n$ when the $k-th$ differential is omitted, as well as $x_{\neq k}$, or $x + z_k$, to indicate that the $k-th$ coordinate in $x \in \mathbb{R}_+^n$ is omitted, or incremented by $z_k \in \mathbb{R}_+$, respectively. Furthermore, we write Γ^n to denote the boundary faces of $\mathbb{R}_+^n$, i.e., $\bigcup_{i=1}^n \{x \in \mathbb{R}_+^n / x_i = 0\}$, and $\mathbb{R}_{>0}^n$ to denote $\mathbb{R}_+^n \setminus \Gamma^n$. Finally, since throughout the paper we will just need to consider the index set $\{1, \ldots, n\}$, from now on we will omit it, writing for example $\forall i$ instead of $\forall i \in \{1, \ldots, n\}$.

Throughout the paper we will assume that the stationary regime exists and that, in this regime, the law of X_t in $\mathbb{R}^n_+$ admits a density (in the sense of distributions), denoted as $p(x)$, $x \in \mathbb{R}^n_+$. Note that since $\forall i$, $a_{i,i}(0_i) > 0$ everywhere in $\mathbb{R}^{n-1}_+$, by (10, Lemma 2.1) applied to the stationary version of semi-martingale (X_t), p does not contain Dirac's delta (or impulse) functions that put probability mass in the boundary faces of $\mathbb{R}^n_+$, i.e., it does not put probability mass in Γ^n. Furthermore, we will assume that the following four conditions are satisfied, $\forall i, j$:

c1) $\int_{\mathbb{R}^{2n}_+} z_i \Phi(x) k(x,z) dz dx = \int_{\mathbb{R}^{n+1}_+} z_i \Phi(x) k_i(x,z_i) dz_i dx < +\infty$, where $k_i(x,z_i)$ denotes the $i-th$ one-dimensional marginal density extracted from $k(x,z)$, $\forall x \in \mathbb{R}^n_+$.

c2) $\int_{\mathbb{R}^n_+} |b_i(x)| p(x) dx < +\infty$.

c3) $\int_{\mathbb{R}^n_+} |a_{i,j}(x)| p(x) dx < +\infty$.

c4) $p(0_i^+)$ exists, a.e. in $\mathbb{R}^{n-1}_+$, and $\int_{\mathbb{R}^{n-1}_+} \sup_{x_i \leq \eta_i} \{a_{i,i}(x) p(x)\} dx_{\neq i} < +\infty$, for some $\eta_i > 0$. Note that if p is separable (i.e., if for a.e. $x \in \mathbb{R}^n_+$, $p(x) = \prod_{k=1}^n p_k(x_k)$ where $\forall k$, p_k is the $k-th$ one-dimensional marginal density extracted from p), and if furthermore $a_{i,i}(x) = a_{i,i}(x_i)$, $\forall x \in \mathbb{R}^n_+$, then this condition reduces to the existence and finiteness of $p_i(0^+)$.

Condition c1) guarantees that hypothesis A (11, p. 173) is satisfied in stationary regime, conditions c2) and c3) allow the use of Fubini's theorem in subsequent proofs in the paper, as well as they guarantee that the Laplace transforms of $b_i p$ and $a_{i,j} p$ exist, $\forall i, j$, and finally, condition c4) allows the use of some previously stated results in (10).

We define the operator $\mathcal{T} : \mathcal{C}^2(\mathbb{R}^n_+) \rightarrow \mathcal{C}(\mathbb{R}^n_+)$, as follows:

$$\mathcal{T} f(x) \triangleq \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

The following lemmas will be used in subsequent sections of the paper.

Lemma 1. *The following relationship holds,* $\forall f \in \mathcal{C}^2_b(\mathbb{R}^n_+)$*:*

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n_+} \mathcal{T} f(x) p(x) dx + \frac{1}{2} \sum_{i,j=1}^{n} \frac{R_{i,j}}{R_{j,j}} \int_{\mathbb{R}^{n-1}_+} \frac{\partial f}{\partial x_i}(0_j) a_{j,j}(0_j) p(0_j^+) dx_{\neq j} \\ &+ \int_{\mathbb{R}^{2n}_+} \{f(x+z) - f(x)\} \Phi(x) k(x,z) dz dx \end{aligned} \qquad (2.2)$$

Proof. Let $f \in \mathcal{C}_b^2(\mathbb{R}_+^n)$. Using Itô's formula (11, Theorem 33, p. 74), equation (2.1) and the fact that $d[X^i, X^j]_s^c = a_{i,j}(X_{s-})ds$, we obtain:

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=1}^n \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) b_i(X_{s-}) ds + \sum_{i,j=1}^n \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) \sigma_{i,j}(X_{s-}) dW_s^j \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_{0+}^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}) a_{i,j}(X_{s-}) ds + \sum_{0<s\le t} \triangle f(X_s) \\ &+ \sum_{i,j=1}^n \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) R_{i,j} dZ_s^j \end{aligned}$$

where $\triangle f(X_s) \triangleq f(X_s) - f(X_{s-})$. Assuming stationary regime and taking expectations to the equation above, we obtain:

$$0 = t \int_{\mathbb{R}_+^n} \mathcal{T} f(x) p(x) dx + \sum_{i,j=1}^n \mathbb{E} \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) R_{i,j} dZ_s^j + \mathbb{E} \sum_{0<s\le t} \triangle f(X_s) \tag{2.3}$$

where we have used that $f \in \mathcal{C}_b^2(\mathbb{R}_+^n)$, the fact that the stochastic integrals involved are centered, continuous local martingales, and conditions c2) and c3) along with Fubini's theorem. Note that all the remaining expectations in equation (2.3) are to be taken in stationary regime. Now, by condition c4) and (10, Theorem 3.2 and Lemma 2.2), we have:

$$\begin{aligned} \sum_{i,j=1}^n \mathbb{E} \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) R_{i,j} dZ_s^j &= \sum_{i,j=1}^n \mathbb{E} \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-}) \frac{R_{i,j}}{2R_{j,j}} dL_s^j \\ &= \frac{t}{2} \sum_{i,j=1}^n \frac{R_{i,j}}{R_{j,j}} \int_{\mathbb{R}_+^{n-1}} \frac{\partial f}{\partial x_i}(0_j) a_{j,j}(0_j) p(0_j^+) dx_{\neq j} \end{aligned}$$

Note that X_{s-} can be replaced by X_s in the r.h.s. of the first equality above since $\forall j$, the random measure induced by (L_s^j) in $\mathbb{R}_+$ is diffuse, and (X_s) is *càdlàg*, P-a.s. Finally, since:

$$\mathbb{E} \sum_{0<s\le t} \triangle f(X_s) = t \int_{\mathbb{R}_+^{2n}} \{f(x+z) - f(x)\} \Phi(x) k(x,z) dz dx$$

equation (2.2) follows after dividing all the previous expressions by $t > 0$. The lemma is now proved. □

From Lemma 1 we can easily obtain the Fokker-Plank equation in the stationary setting, as stated in the next corollary.

Corollary 1. *Assume that* $\forall i,j$, $b_i p \in \mathcal{C}^1(\mathbb{R}^n)$, $a_{i,j}p \in \mathcal{C}^2(\mathbb{R}^n)$, *and the n-dimensional Laplace transform of* $\frac{\partial(a_{i,i}p)}{\partial x_i}$ *exists. Then,* $\forall x \in \mathbb{R}^n_+$ *we have:*

$$0 = -\sum_{i=1}^{n} \frac{\partial(b_i p)}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2(a_{i,j}p)}{\partial x_i \partial x_j}(x) + \int_0^{x_1} \cdots \int_0^{x_n} \Phi(\xi) k(\xi, x-\xi) d\xi - \Phi(x) \quad (2.4)$$

Proof. From Lemma 1, and denoting as $< \cdot, \cdot >$ the usual inner product in $\mathbb{R}^n$, by considering $f(x) = \exp\{- < \nu, x >\}$ with $\nu \in \mathbb{R}^n_{>0}$, which belongs to $\mathcal{C}^2_b(\mathbb{R}^n_+)$, we obtain:

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\sum_{i=1}^{n} \frac{\nu_i \mathcal{L}_n[a_{i,i}p](\nu) - \mathcal{L}_{n-1}[a_{i,i}(0_i)p(0_i^+)](\nu_{\neq i})}{\prod_{\substack{k=1\\k\neq i}}^{n} \nu_k} + \frac{\mathcal{L}_n[h](\nu)}{\prod_{k=1}^{n}\nu_k} - \frac{\mathcal{L}_n[\Phi](\nu)}{\prod_{k=1}^{n}\nu_k} \\ & - \sum_{i=1}^{n} \frac{\mathcal{L}_n[b_i p](\nu)}{\prod_{\substack{k=1\\k\neq i}}^{n}\nu_k} - \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{R_{i,j}}{R_{j,j}} \frac{\mathcal{L}_{n-1}[a_{j,j}(0_j)p(0_j^+)](\nu_{\neq j})}{\prod_{\substack{k=1\\k\neq i,j}}^{n}\nu_k} \frac{1}{\nu_j} + \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\mathcal{L}_n[a_{i,j}p](\nu)}{\prod_{\substack{k=1\\k\neq i,j}}^{n}\nu_k} \end{aligned}$$

where $\forall x \in \mathbb{R}^n_+$, $h(x) \triangleq \int_0^{x_1} \cdots \int_0^{x_n} \Phi(\xi)k(\xi, x-\xi)d\xi$. By taking (n-dimensional) inverse Laplace transform to the equation above, and $\frac{\partial^n}{\partial x_1 \cdots \partial x_n}$ to the resulting equation, the corollary follows. □

3. Stationary Equations for the Separable Case

In this section we specialize all the previous results, as well as we obtain some new ones, for the case where the stationary density p is separable, i.e., it can be expressed as the product of its one-dimensional marginals. These results will lead us to necessary and sufficient conditions for this factorization to be possible, as it will be stated in Theorem 1 in the next section of the paper.

From now on we will assume, in addition of course to all the assumptions already stated, the following:

- $\forall i,j$ and $\forall x \in \mathbb{R}^n_+$, $b_i(x) = b_i(x_i)$ and $a_{i,j}(x) = a_{i,j}(x_i, x_j)$. Note that $a_{i,j}(x) = a_{i,j}(x_i, x_j)$, $\forall i,j$ and $\forall x \in \mathbb{R}^n_+$, if, for example, $\sigma_{i,j}(x) = \sigma_{i,j}(x_i)$, $\forall i,j$ and $\forall x \in \mathbb{R}^n_+$.

- $\forall x \in \mathbb{R}^n_+$, the density $k(x, z)$ is separable, taking the form $\prod_{i=1}^n k_i(x_i, z_i)$. That is, $\forall x \in \mathbb{R}^n_+$, the density $k(x, z)$ is written as the product of its one-dimensional marginals and, in addition, its $i - th$ one-dimensional marginal depends on the parameter x only through its $i - th$ coordinate x_i, $\forall i$.
- The intensity λ is independent of (X_{t-}) and takes the form $\sum_{i=1}^n \lambda_i$, where $\forall i$, λ_i is the intensity associated with the jumps in the $i - th$ coordinate of (X_t), and the λ_i's are independent.

Remark 1. Note that, since the jumps associated with different coordinates of (X_t) are independent, two or more of its coordinates do not jump simultaneously at any given time t, P-a.s., and since furthermore $\forall j$, λ_j is independent of (X^j_{t-}) and $a_{j,j}(0_j) = a_{j,j}(0)$, equations (2.2) in Lemma 1 and (2.4) in Corollary 1 now take the form:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n_+} \mathcal{T} f(x)p(x)dx + \frac{1}{2}\sum_{i,j=1}^n \frac{R_{i,j}}{R_{j,j}} a_{j,j}(0) \int_{\mathbb{R}^{n-1}_+} \frac{\partial f}{\partial x_i}(0_j)p(0^+_j)dx_{\neq j} \\ &+ \sum_{i=1}^n \mathbb{E}\lambda_i \int_{\mathbb{R}^{n+1}_+} \{f(x+z_i) - f(x)\}p(x)k_i(x_i, z_i)dz_i dx \end{aligned} \tag{3.1}$$

and:

$$\begin{aligned} 0 &= -\sum_{i=1}^n \frac{\partial(b_i p)}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2(a_{i,j}p)}{\partial x_i \partial x_j}(x) \\ &+ \sum_{i=1}^n \mathbb{E}\lambda_i \left\{ \int_0^{x_i} p(\xi_i, x_{\neq i})k_i(\xi_i, x_i - \xi_i)d\xi_i - p(x) \right\} \end{aligned} \tag{3.2}$$

respectively, where $\forall i$, $\mathbb{E}\lambda_i$ in equations (3.1) and (3.2) denotes the expected value of the intensity λ_i, and $p(\xi_i, x_{\neq i})$ in equation (3.2) denotes $p(x_1, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_n)$, i.e., the stationary density p when its $i - th$ coordinate x_i has been replaced by the integration variable ξ_i. Moreover, equation (3.1) holds $\forall f \in \mathcal{C}^2_b(\mathbb{R}^n_+)$.

Remark 2. Note that, if $\widetilde{p}$ is another density in $\mathbb{R}^n_+$ satisfying equation (3.1) $\forall f \in \mathcal{C}^2_b(\mathbb{R}^n_+)$, then $\widetilde{p}(x) = p(x)$ for a.e. $x \in \mathbb{R}^n_+$. Moreover, this same conclusion holds true if $\widetilde{p}$ is another density in $\mathbb{R}^n_+$ satisfying equation (3.1) $\forall f = \prod_{i=1}^n f_i$, where $\forall i$, f_i is any function in $\mathcal{C}^2_b(\mathbb{R}_+)$.

Furthermore, $\forall i$ we now define the operator $\mathcal{T}_i : \mathcal{C}^2(\mathbb{R}_+) \to \mathcal{C}(\mathbb{R}_+)$, as follows:

$$\mathcal{T}_i f(x) \triangleq b_i(x_i)\frac{df}{dx_i}(x_i) + a_{i,i}(x_i)\frac{d^2 f}{dx_i^2}(x_i)$$

Corollary 2. *Assume the stationary density p is separable, i.e., for a.e. $x \in \mathbb{R}_+^n$, $p(x) = \prod_{i=1}^n p_i(x_i)$, where the family of one-dimensional marginals, $\{p_i\}_{i=1}^n$ is such that $\forall i$, p_i does not put probability mass at the origin and satisfies conditions c1), c2) and c4), and $\forall i, j$, p_i and p_j satisfy condition c3). Then $\forall i$, p_i satisfies the following relationship, $\forall f \in \mathcal{C}_b^2(\mathbb{R}_+)$:*

$$\begin{aligned} 0 &= \frac{1}{2}\frac{df}{dx_i}(0)a_{i,i}(0)p_i(0^+) + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} \{f(x_i+z_i) - f(x_i)\}p_i(x_i)k_i(x_i,z_i)dz_i dx_i \\ &+ \int_{\mathbb{R}_+} \mathcal{T}_i f(x_i)p_i(x_i)dx_i + \frac{1}{2}\int_{\mathbb{R}_+} \frac{df}{dx_i}(x_i)p_i(x_i)dx_i \sum_{\substack{j=1 \\ j\neq i}}^n \frac{R_{i,j}}{R_{j,j}} a_{j,j}(0)p_j(0^+) \end{aligned} \tag{3.3}$$

Proof. Follows in a straightforward manner from equation (3.1) by taking, for each i, $f(x_i) \in \mathcal{C}_b^2(\mathbb{R}_+)$, and using the product-form of the stationary density p. □

Lemma 2. *Under the same assumptions as in Corollary 2, $\forall i$ we have:*

$$-\frac{1}{2}\sum_{j=1}^n \frac{R_{i,j}}{R_{j,j}} a_{j,j}(0)p_j(0^+) = \int_{\mathbb{R}_+} b_i(x_i)p_i(x_i)dx_i + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(x_i)k_i(x_i,z_i)dz_i dx_i \tag{3.4}$$

Proof. First, for each i, apply Meyer-Itô's formula (11, Theorem 51, p.167) to the convex function $f(x_i) = (x_i)^+ = \max\{0, x_i\}$ and use equation (2.1). Then, in stationary regime, by taking expectations and using the fact that the stochastic integrals involved are centered, continuous local martingales, and conditions c2) and c4) along with Fubini's theorem and (10, Theorem 3.2 and Lemma 2.2), we obtain:

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}_+} \mathbf{1}\{x_i > 0\}p_i(x_i)dx_i \sum_{\substack{j=1 \\ j\neq i}}^n \frac{R_{i,j}}{R_{j,j}} a_{j,j}(0)p_j(0^+) &= \frac{1}{2}a_{i,i}(0)p_i(0^+) \\ + \int_{\mathbb{R}_+} \mathbf{1}\{x_i > 0\}b_i(x_i)p_i(x_i)dx_i &+ \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(x_i)k_i(x_i,z_i)dz_i dx_i \end{aligned}$$

where $\mathbf{1}\{\cdot\}$ denotes the usual indicator function of the corresponding event in parenthesis. Since $\forall i$, p_i does not put mass at the origin, the lemma now follows. □

Remark 3. Note that, since R is non-singular, from Lemma 2 we obtain that $p_i(0^+) > 0$, $\forall i$, if and only if $R^{-1}\Lambda < 0$ (componentwise), where Λ is the column vector whose $i-th$ entry is given by $\int_{\mathbb{R}_+} b_i(x_i)p_i(x_i)dx_i + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(x_i)k_i(x_i,z_i)dz_i dx_i$, i.e., if and only if the net-drift in each dimension is strictly negative. This gives us a stability condition, as it will be seen in Section 5.

Corollary 3. *Under the same assumptions as in Corollary 2, $\forall i$ and $\forall f \in \mathcal{C}_b^2(\mathbb{R}_+)$, p_i satisfies:*

$$\begin{aligned} 0 &= \int_{\mathbb{R}_+} \mathcal{T}_i f(x_i)p_i(x_i)dx_i + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} \{f(x_i+z_i) - f(x_i)\}p_i(x_i)k_i(x_i,z_i)dz_i dx_i \\ &- \int_{\mathbb{R}_+} \frac{df}{dx_i}(x_i)p_i(x_i)dx_i \Big\{ \frac{1}{2}a_{i,i}(0)p_i(0^+) + \int_{\mathbb{R}_+} b_i(x_i)p_i(x_i)dx_i \\ &+ \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(x_i)k_i(x_i,z_i)dz_i dx_i \Big\} + \frac{1}{2}\frac{df}{dx_i}(0)a_{i,i}(0)p_i(0^+) \end{aligned} \tag{3.5}$$

Proof. Follows in a straightforward manner by combining the results of Corollary 2 and Lemma 2. □

Remark 4. Note that, as in Remark 2 for p, Corollaries 2 and 3 uniquely characterize each one-dimensional marginal density p_i extracted from p, in the separable setting and in an a.e. sense, of course.

Lemma 3. *Under the same assumptions as in Corollary 2, and if furthermore $\forall i$, $a_{i,i}p_i \in \mathcal{C}^1(\mathbb{R}_+)$ such that the Laplace transform of $\frac{d(a_{i,i}p_i)}{dx_i}$ exists, then $\forall i$, p_i satisfies equation (3.5), $\forall f \in \mathcal{C}_b^2(\mathbb{R}_+)$, if and only if it satisfies, $\forall x_i \in \mathbb{R}_+$:*

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d(a_{i,i}p_i)}{dx_i}(x_i) + \mathbb{E}\lambda_i \Big\{ \int_0^{x_i}\int_0^u p_i(\xi)k_i(\xi, u-\xi)d\xi du - \int_0^{x_i} p_i(\xi)d\xi \Big\} \\ &+ p_i(x_i)\Big\{ \int_{\mathbb{R}_+} b_i(\xi)p_i(\xi)d\xi - b_i(x_i) + \frac{1}{2}a_{i,i}(0)p_i(0^+) \\ &+ \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(\xi)k_i(\xi,z_i)dz_i d\xi \Big\} \end{aligned} \tag{3.6}$$

Proof. The only if part follows in a straightforward manner from Corollary 3 by considering, for each i, $f(x_i) = \exp\{-\nu_i x_i\}$ with $\nu_i > 0$, which belong to $\mathcal{C}_b^2(\mathbb{R}_+)$, and then taking inverse Laplace transform. For the if part, consider, for each i, $f(x_i) \in \mathcal{C}_b^2(\mathbb{R}_+)$. Then, by multiplying equation (3.6) by $-\frac{df}{dx_i}$, integrating over $\mathbb{R}_+$, and using the facts that $\forall i$, p_i does not put mass at the origin and that, by condition c3) with $i = j$, $a_{i,i}(x_i)p_i(x_i) \to 0$ as $x_i \to +\infty$, we obtain equation (3.5). The lemma is now proved. $\Box$

Remark 5. Assuming that $\forall i$, p_i and $b_ip_i \in \mathcal{C}^1(\mathbb{R}_+)$, $a_{i,i}p_i \in \mathcal{C}^2(\mathbb{R}_+)$, and that the Laplace transform of $\frac{d(a_{i,i}p_i)}{dx_i}$ exists, then, by taking $\frac{d}{dx_i}$ to equation (3.6) we obtain the Fokker-Plank equation for each coordinate in the case of stationary regime and product-form density, namely:

$$\begin{aligned} 0 &= \frac{dp_i}{dx_i}(x_i)\left\{\int_{\mathbb{R}_+} b_i(\xi)p_i(\xi)d\xi + \frac{1}{2}a_{i,i}(0)p_i(0^+) + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_ip_i(\xi)k_i(\xi, z_i)dz_id\xi\right\} \\ &- \mathbb{E}\lambda_ip_i(x_i) - \frac{d(b_ip_i)}{dx_i}(x_i) + \frac{1}{2}\frac{d^2(a_{i,i}p_i)}{dx_i^2}(x_i) + \mathbb{E}\lambda_i \int_0^{x_i} p_i(\xi)k_i(\xi, x_i - \xi)d\xi \end{aligned}$$

Remark 6. Note that, if $\forall i$ and $\forall x_i \in \mathbb{R}_+$, $b_i(x_i) = b_i$ (constant) and $k_i(x_i, z_i) = k_i(z_i)$ (independent of x_i), then equation (3.6) has no anticipative terms (in space) and it can suitable be solved numerically, along with the in this case known limit boundary conditions, $\{p_i(0^+)\}_{i=1}^n$ given by equation (3.4) (see also Remark 3).

4. The Main Result: Necessary and Sufficient Conditions

Using the results of the previous sections, we are now in position to state and prove the main result of the paper concerning necessary and sufficient conditions for product-form distributions in stationary regime, assuming this regime and its corresponding density p, satisfying conditions c1) to c4), exist (note we have not shown that the existence of solution for equation (3.1) is sufficient for the existence of stationary regime).

Theorem 1. *Let $\{p_i\}_{i=1}^n$ be the family of one-dimensional marginals extracted from the stationary density p. Then, p is separable, i.e., $p(x) = \prod_{i=1}^n p_i(x_i)$ for a.e. $x \in \mathbb{R}_+^n$, if and only if $\forall i, j$, $i \neq j$, we have:*

$$a_{i,j}(x_i, x_j) = g_{i,j}(x_i) + g_{j,i}(x_j) \tag{4.1}$$

for $F_i \times F_j$*-a.e.* $(x_i, x_j) \in \mathbb{R}_+^2$*, where* $\forall k$*,* F_k *denotes the corresponding probability distribution in* $\mathbb{R}_+$ *associated with* p_k*, and* $\forall k, l$*,* $k \neq l$*, the tuple* $(g_{k,l}, g_{l,k})$ *is defined,* $F_k \times F_l$*-a.e., by:*

$$g_{k,l}(x_k) \triangleq \frac{R_{l,k}}{2R_{k,k}} \frac{a_{k,k}(0)p_k(0^+)}{p_k(x_k)} \int_{x_k}^{+\infty} p_k(\xi)d\xi \tag{4.2}$$

and the corresponding expression for $g_{l,k}(x_l)$ obtained by interchanging k and l above.

Proof. Assume that p is separable. Then, using equation (3.1) with $f = \prod_{i=1}^n f_i$, where $\forall i$, f_i is any function in $\mathcal{C}_b^2(\mathbb{R}_+)$, and equation (3.3), we obtain:

$$0 = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \Theta_{i,j}(f_i, f_j) \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \int_{\mathbb{R}_+} f_k(x_k)p_k(x_k)dx_k \tag{4.3}$$

where:

$$\Theta_{i,j}(f_i, f_j) \triangleq \Gamma_{i,j}(f_i, f_j) + \int_{\mathbb{R}_+^2} \frac{df_i}{dx_i}(x_i) \frac{df_j}{dx_j}(x_j) a_{i,j}(x_i, x_j) p_i(x_i) p_j(x_j) dx_i dx_j$$

and:

$$\Gamma_{i,j}(f_i, f_j) \triangleq \frac{R_{i,j}}{R_{j,j}} a_{j,j}(0)p_j(0^+) \int_{\mathbb{R}_+} \frac{df_i}{dx_i}(x_i)p_i(x_i)dx_i \left\{ f_j(0) - \int_{\mathbb{R}_+} f_j(x_j)p_j(x_j)dx_j \right\}$$

Now, equation (4.3) holds for any f of the considered form if and only if $\Theta_{i,j}(f_i, f_j) + \Theta_{j,i}(f_j, f_i) = 0$, $\forall i, j$, $i \neq j$, and $\forall f_i$ and $f_j \in \mathcal{C}_b^2(\mathbb{R}_+)$. Thus, for $i \neq j$, $a_{i,j}$ must be such that, $\forall f_i$ and $f_j \in \mathcal{C}_b^2(\mathbb{R}_+)$:

$$2 \int_{\mathbb{R}_+^2} \frac{df_i}{dx_i}(x_i) \frac{df_j}{dx_j}(x_j) a_{i,j}(x_i, x_j) p_i(x_i) p_j(x_j) dx_i dx_j = \Gamma_{i,j}(f_i, f_j) + \Gamma_{j,i}(f_j, f_i) \tag{4.4}$$

where we have used the fact that the diffusion matrix a is symmetric. Thinking of the above expression as an equation for $a_{i,j}$, with $i \neq j$, we immediately obtain uniqueness in the $F_i \times F_j$-a.e. sense. Therefore, for $i \neq j$, we may look for a solution of the form $a_{i,j}(x_i, x_j) = g_{i,j}(x_i) + g_{j,i}(x_j)$. Inserting this in equation (4.4), and considering

$f_k(x_k) = \exp\{-\nu_k x_k\}$ with $\nu_k > 0$, $k = i, j$, which belong to $\mathcal{C}_b^2(\mathbb{R}_+)$, we obtain:

$$\begin{aligned}\mathcal{L}[g_{i,j}p_i](\nu_i)\mathcal{L}[p_j](\nu_j) \quad &+ \quad \mathcal{L}[g_{j,i}p_j](\nu_j)\mathcal{L}[p_i](\nu_i) \\ &= \frac{R_{i,j}}{2R_{j,j}} a_{j,j}(0)p_j(0^+)\left\{\frac{1}{\nu_j} - \frac{\mathcal{L}[p_j](\nu_j)}{\nu_j}\right\}\mathcal{L}[p_i](\nu_i) \\ &+ \frac{R_{j,i}}{2R_{i,i}} a_{i,i}(0)p_i(0^+)\left\{\frac{1}{\nu_i} - \frac{\mathcal{L}[p_i](\nu_i)}{\nu_i}\right\}\mathcal{L}[p_j](\nu_j)\end{aligned}$$

Since the above equation must hold $\forall \nu_i$ and $\nu_j > 0$, we conclude that:

$$\mathcal{L}[g_{i,j}p_i](\nu_i) = \frac{R_{j,i}}{2R_{i,i}} a_{i,i}(0)p_i(0^+)\left\{\frac{1}{\nu_i} - \frac{\mathcal{L}[p_i](\nu_i)}{\nu_i}\right\} \tag{4.5}$$

and the corresponding expression for $\mathcal{L}[g_{j,i}p_j](\nu_j)$ by interchanging i and j in this last equation. Hence, equations (4.1) and (4.2) now follow. For the converse, assume that $\forall i, j$, $i \neq j$, $a_{i,j}(x_i, x_j)$ satisfies equation (4.1) for $F_i \times F_j$-a.e. $(x_i, x_j) \in \mathbb{R}_+^2$, with $g_{i,j}$ and $g_{j,i}$ as in (4.2). Since $\forall i$ and $\forall f_i \in \mathcal{C}_b^2(\mathbb{R}_+)$ we have:

$$\begin{aligned}\int_{\mathbb{R}_+} \frac{df_i}{dx_i}(x_i)\left(\int_{x_i}^{+\infty} p_i(\xi)d\xi\right)dx_i \quad &= \quad \lim_{x_i \to +\infty} f_i(x_i)\int_{x_i}^{+\infty} p_i(\xi)d\xi - f_i(0) \\ &+ \int_{\mathbb{R}_+} f_i(x_i)p_i(x_i)dx_i \\ &= -f_i(0) + \int_{\mathbb{R}_+} f_i(x_i)p_i(x_i)dx_i\end{aligned}$$

then $\Theta_{i,j}(f_i, f_j) + \Theta_{j,i}(f_j, f_i) = 0$, $\forall i, j$, $i \neq j$, and $\forall f_i$ and $f_j \in \mathcal{C}_b^2(\mathbb{R}_+)$. Therefore, equation (3.1) is satisfied with $\prod_{i=1}^n p_i$ in place of p, $\forall f = \prod_{i=1}^n f_i$, where $\forall i$, f_i is any function in $\mathcal{C}_b^2(\mathbb{R}_+)$. The theorem now follows (see Remark 2). □

Remark 7. Note that in the statement of Theorem 1, no assumptions regarding the existence of derivatives involving b, a or p are needed.

Remark 8. In the case where there are no jumps and the drift vector $b(\cdot)$ is constant, by considering $f_i(x_i) = \exp\{-\nu_i x_i\} \in \mathcal{C}_b^2(\mathbb{R}_+)$ $(\nu_i > 0)$, for each i, from equation (3.5) we obtain:

$$a_{i,i}(0)p_i(0^+)\left\{\frac{1}{\nu_i} - \frac{\mathcal{L}[p_i](\nu_i)}{\nu_i}\right\} = \mathcal{L}[a_{i,i}p_i](\nu_i)$$

and therefore, using equation (4.5), the separability condition can be written in this case as:

$$a_{i,j}(x_i, x_j) = \frac{R_{j,i}}{2R_{i,i}} a_{i,i}(x_i) + \frac{R_{i,j}}{2R_{j,j}} a_{j,j}(x_j) \tag{4.6}$$

$\forall i, j,\ i \neq j$, and for $F_i \times F_j$-a.e. $(x_i, x_j) \in \mathbb{R}_+^2$ (i.e., as it will be seen in Section 5.1, Subsection 5.1.1, and under the conditions thereon, for a.e. $(x_i, x_j) \in \mathbb{R}_+^2$). Note that this generalizes the separability condition stated in (5) for the simpler case of SRBM, where the coefficients in b and a do not depend on x and there are no jumps. Finally, note that from equation (4.6), condition c3) is satisfied $\forall i, j$ once satisfied $\forall i = j$.

From Theorem 1 we conclude that product-form stationary distributions are only possible under very special conditions. In fact, we have the following negative result.

Corollary 4. *If the diffusion matrix a has constant off-diagonal elements, then a product-form stationary density is not possible unless: Either the jumps in each coordinate are identically null and the drift b, as well as the diagonal elements in a, are constant; Or a and the reflection matrix R are diagonal.*

Proof. Let $\{p_i\}_{i=1}^n$ be the family of one-dimensional marginals extracted from the stationary density p, and assume that $\forall i, j,\ i \neq j$, $a_{i,j}(x_i, x_j) = a_{i,j}$ (constant). Then, from Theorem 1 product-form stationary density is only possible in the following two cases. First, when each p_i is such that $\frac{1}{p_i(x_i)} \int_{x_i}^{+\infty} p_i(\xi) d\xi = \frac{1}{p_i(0^+)}$, $\forall x_i \in \mathbb{R}_+$. But then $\forall i$, p_i is absolutely continuous in $\mathbb{R}_+$ (i.e., it is continuous and a.e. differentiable in $\mathbb{R}_+$). Hence, $p_i(x_i) = p_i(0^+) \exp\{-p_i(0^+) x_i\}$, $\forall i$ and $\forall x_i \in \mathbb{R}_+$, i.e., each p_i corresponds to an exponential density with parameter $p_i(0^+)$, in which case, from Corollary 3 the jumps in each coordinate must be identically null and the drift b, as well as the diagonal elements in a, must be constant. Note that this case corresponds to SRBM. Second, when $\forall i, j,\ i \neq j$, $a_{i,j} = 0$ (i.e., a is diagonal), and either $R_{i,j} = 0$ or $p_j(0^+) = 0$. But then, from Lemma 2 we conclude that, $\forall i$:

$$0 = \frac{1}{2} a_{i,i}(0) p_i(0^+) + \int_{\mathbb{R}_+} b_i(x_i) p_i(x_i) dx_i + \mathbb{E}\lambda_i \int_{\mathbb{R}_+^2} z_i p_i(x_i) k_i(x_i, z_i) dz_i dx_i$$

and therefore, from Corollary 3, if $p_i(0^+) = 0$ for some i, then $p_i(x_i) = 0$ for a.e. $x_i \in \mathbb{R}_+$, which can not be. Thus, we conclude that R must be diagonal as well. The corollary is now proved. □

Remark 9. Note that Corollary 4 generalizes the negative results, regarding product-form stationary distributions, shown for the case of n-dimensional reflected Lévy processes in (8).

5. Some Examples

We conclude the paper with a presentation of some examples where explicit computations are possible.

5.1. Continuous Case

We first discuss some examples in the continuous case, i.e., in the absence of jumps. Then, assuming the hypotheses of Lemma 3 are satisfied and that, in addition, $\forall i$ and $\forall x_i \in \mathbb{R}_+$, $a_{i,i}(x_i) > 0$, from equation (3.6) we obtain:

$$p_i(x_i) = \frac{a_{i,i}(0)p_i(0^+)}{a_{i,i}(x_i)} \exp\left\{-2\int_0^{x_i} \frac{\mathbb{E}b_i - b_i(\xi) + \frac{1}{2}a_{i,i}(0)p_i(0^+)}{a_{i,i}(\xi)} d\xi\right\} \tag{5.1}$$

$\forall i$ and $\forall x_i \in \mathbb{R}_+$, where $\mathbb{E}b_i = \int_{\mathbb{R}_+} b_i(\zeta)p_i(\zeta)d\zeta$. We can find all the unknowns involved in equation (5.1) in the following two cases.

5.1.1. *Constant Drift Vector b* In this case, equation (5.1) reduces to:

$$p_i(x_i) = \frac{a_{i,i}(0)p_i(0^+)}{a_{i,i}(x_i)} \exp\left\{-a_{i,i}(0)p_i(0^+)\int_0^{x_i} \frac{d\xi}{a_{i,i}(\xi)}\right\} \tag{5.2}$$

Furthermore, from Lemma 2 we have:

$$R\gamma_0 = -2b$$

where b is the constant drift vector and γ_0 is the vector whose $i-th$ entry is given by $\frac{a_{i,i}(0)}{R_{i,i}}p_i(0^+)$. Then, since R is non-singular, the limit boundary conditions, $\{p_i(0^+)\}_{i=1}^n$ are uniquely determined by inverting the above system of linear equations. Of course, we require that $\gamma_0 > 0$ (componentwise), which imposes the stability condition $R^{-1}b < 0$ (componentwise), i.e., the net-drift in each dimension must be strictly negative.

Furthermore, if $\forall i$:

$$\int_{\mathbb{R}_+} \frac{dx_i}{a_{i,i}(x_i)} = +\infty$$

then each p_i integrates to the unity. Note that, if $a_{i,i}$ is constant, then the above condition is trivially satisfied and, from equation (5.2), p_i corresponds to an exponential density with parameter $p_i(0^+)$. Moreover, note that from equation (5.2):

$$\frac{d(a_{i,i}p_i)}{dx_i}(x_i) = -a_{i,i}(0)p_i(0^+)p_i(x_i)$$

and therefore, the Laplace transform of $\frac{d(a_{i,i}p_i)}{dx_i}$ exists, $\forall i$. In addition, if $\forall i$:

$$\int_{\mathbb{R}_+} \exp\left\{-a_{i,i}(0)p_i(0^+)\int_0^{x_i} \frac{d\xi}{a_{i,i}(\xi)}\right\}dx_i < +\infty$$

then condition c3) is satisfied, $\forall i = j$ (see Remark 8). Finally, as it was already mentioned in Remark 8, the separability condition in this case reduces to:

$$a_{i,j}(x_i, x_j) = \frac{R_{j,i}}{2R_{i,i}}a_{i,i}(x_i) + \frac{R_{i,j}}{2R_{j,j}}a_{j,j}(x_j)$$

$\forall i, j$, $i \neq j$, and for $F_i \times F_j$-a.e. $(x_i, x_j) \in \mathbb{R}^2_+$, i.e., for a.e. $(x_i, x_j) \in \mathbb{R}^2_+$ since, from equation (5.2), we conclude that F_i is equivalent to Lebesgue measure in $\mathbb{R}_+$, $\forall i$. Thus, under all the previous requirements, if the stationary setting and its corresponding density p, satisfying conditions c3) and c4), exist, then:

$$p(x) = \prod_{i=1}^{n} \frac{a_{i,i}(0)p_i(0^+)}{a_{i,i}(x_i)} \exp\left\{-a_{i,i}(0)p_i(0^+)\int_0^{x_i} \frac{d\xi}{a_{i,i}(\xi)}\right\}$$

for a.e. $x \in \mathbb{R}^n_+$. Note that this generalizes the exponential product-form density obtained in the case of SRBM, as shown for example in (5) or (1).

5.1.2. *Normal Reflections* In this case the reflection matrix R is diagonal and therefore, using Lemma 2, equation (5.1) reduces to:

$$p_i(x_i) = \frac{a_{i,i}(0)p_i(0^+)}{a_{i,i}(x_i)} \exp\left\{2\int_0^{x_i} \frac{b_i(\xi)}{a_{i,i}(\xi)}d\xi\right\} \tag{5.3}$$

From the normalization condition we obtain:

$$p_i(0^+) = \left\{ a_{i,i}(0) \int_{\mathbb{R}_+} \frac{\phi_i(x_i)}{a_{i,i}(x_i)} dx_i \right\}^{-1}$$

where:

$$\phi_i(x_i) \triangleq \exp\left\{ 2 \int_0^{x_i} \frac{b_i(\xi)}{a_{i,i}(\xi)} d\xi \right\}$$

and therefore, we require that $\forall i$:

$$\int_{\mathbb{R}_+} \frac{\phi_i(x_i)}{a_{i,i}(x_i)} dx_i < +\infty$$

Furthermore, note that $\forall i$, equation (3.4) in Lemma 2 is back verified if $b_i(x_i) < 0$, for a.e. $x_i \in \mathbb{R}_+$, and:

$$\int_{\mathbb{R}_+} \frac{b_i(x_i)}{a_{i,i}(x_i)} dx_i = -\infty$$

Also, note that the above requirements guarantee that condition c2) is satisfied, $\forall i$. Moreover, note that from equation (5.3):

$$\frac{d(a_{i,i} p_i)}{dx_i}(x_i) = 2 b_i(x_i) p_i(x_i)$$

and therefore, under the previous requirements, the Laplace transform of $\frac{d(a_{i,i} p_i)}{dx_i}$ exists, $\forall i$. In addition, if $\forall i$:

$$\int_{\mathbb{R}_+} \exp\left\{ 2 \int_0^{x_i} \frac{b_i(\xi)}{a_{i,i}(\xi)} d\xi \right\} dx_i < +\infty$$

then condition c3) is satisfied, $\forall i = j$. Finally, the separability condition in this case reduces, of course, to:

$$a_{i,j}(x_i, x_j) = 0$$

$\forall i, j,\ i \neq j$, and for $F_i \times F_j$-a.e. $(x_i, x_j) \in \mathbb{R}_+^2$, i.e., for a.e. $(x_i, x_j) \in \mathbb{R}_+^2$ since, from equation (5.3), we conclude that F_i is equivalent to Lebesgue measure in $\mathbb{R}_+$, $\forall i$. Thus, under all the previous requirements, if the stationary setting and its corresponding density p, satisfying conditions c2), c3) and c4), exist, then:

$$p(x) = \prod_{i=1}^{n} \frac{a_{i,i}(0)p_i(0^+)}{a_{i,i}(x_i)} \exp\left\{ 2\int_0^{x_i} \frac{b_i(\xi)}{a_{i,i}(\xi)} d\xi \right\}$$

for a.e. $x \in \mathbb{R}_+^n$.

5.2. *Càdlàg* Case

We now allow a non-identically null jump measure. Then, assuming that $\forall i$ and $\forall x_i \in \mathbb{R}_+$, $b_i(x_i) = b_i$ (constant), $a_{i,i}(x_i) = a_{i,i} > 0$ (constant) and $k_i(x_i, z_i) = k_i(z_i)$ (independent of x_i), by considering $f_i(x_i) = \exp\{-\nu_i x_i\} \in \mathcal{C}_b^2(\mathbb{R}_+)$ ($\nu_i > 0$), for each i, from equation (3.5) we obtain:

$$\mathcal{L}[p_i](\nu_i) = \frac{\frac{1}{2}a_{i,i}p_i(0^+)}{\frac{1}{2}a_{i,i}p_i(0^+) + \mathbb{E}\lambda_i\mathbb{E}k_i + \frac{1}{2}a_{i,i}\nu_i + \frac{\mathbb{E}\lambda_i}{\nu_i}\{\mathcal{L}[k_i](\nu_i) - 1\}} \tag{5.4}$$

where $\mathbb{E}k_i = \int_{\mathbb{R}_+} z_i k_i(z_i) dz_i$. Of course, we ask $\mathbb{E}\lambda_i\mathbb{E}k_i < +\infty$, $\forall i$. Furthermore, from Lemma 2 we have:

$$R\gamma_0 = -2\beta \tag{5.5}$$

where γ_0 is as in Section 5.1.1 and β is the vector whose $i-th$ entry is given by $b_i + \mathbb{E}\lambda_i\mathbb{E}k_i$. Then, since R is non-singular, the limit boundary conditions, $\{p_i(0^+)\}_{i=1}^n$ are uniquely determined by inverting the above system of linear equations. As in Section 5.1.1, we require that $\gamma_0 > 0$ (componentwise), i.e., the net-drift in each dimension must be strictly negative. Thus, under the previous requirements, if the stationary setting and its corresponding density p, satisfying conditions c3) and c4), exist, and if the separability condition in Theorem 1 and condition c3) are satisfied by the family $\{p_i\}_{i=1}^n$ obtained from equation (5.4) (note that this last condition is trivially satisfied when $i = j$ since a has constant diagonal), then:

$$p(x) = \prod_{i=1}^{n} p_i(x_i)$$

for a.e. $x \in \mathbb{R}_+^n$.

5.2.1. *Exponentially Distributed Jumps* As a particular example of the results above, we now consider the case of exponentially distributed jumps, i.e., when k_i corresponds to an exponential density with parameter $\theta_i > 0$, $\forall i$. Then, by taking inverse Laplace transform to equation (5.4) we obtain, $\forall i$ and $\forall x_i \in \mathbb{R}_+$:

$$p_i(x_i) = \frac{p_i(0^+)}{2}\{q_i(x_i) + r_i(x_i)\} \tag{5.6}$$

where:

$$\begin{aligned} q_i(x_i) &\triangleq \left(1 - \frac{M_i}{N_i}\right)\exp\{-(\theta_i + M_i - N_i)x_i\} \\ r_i(x_i) &\triangleq \left(1 + \frac{M_i}{N_i}\right)\exp\{-(\theta_i + M_i + N_i)x_i\} \end{aligned}$$

and:

$$\begin{aligned} 2M_i &\triangleq p_i(0^+) - \theta_i + \frac{2\mathbb{E}\lambda_i}{a_{i,i}\theta_i} \\ N_i &\triangleq \sqrt{\frac{2\mathbb{E}\lambda_i}{a_{i,i}} + M_i^2} \geq |M_i| \end{aligned}$$

Then, since $\forall i$:

$$\theta_i + M_i = \frac{p_i(0^+)}{2} + \frac{\theta_i}{2} + \frac{\mathbb{E}\lambda_i}{a_{i,i}\theta_i}$$

and:

$$N_i = \sqrt{\left\{\frac{p_i(0^+)}{2} + \frac{\theta_i}{2} + \frac{\mathbb{E}\lambda_i}{a_{i,i}\theta_i}\right\}^2 - p_i(0^+)\theta_i}$$

equation (5.6) gives a valid density in $\mathbb{R}_+$ as long as $p_i(0^+) > 0$ (see equation (5.5) and the comments that follow it). In addition, since $\forall i$, $\int_{\mathbb{R}_+} x_i p_i(x_i) dx_i < +\infty$, condition c3) is satisfied, $\forall i, j$, in the separable case. Finally, the separability condition is:

$$a_{i,j}(x_i, x_j) = g_{i,j}(x_i) + g_{j,i}(x_j)$$

$\forall i, j$, $i \neq j$, and for a.e. $(x_i, x_j) \in \mathbb{R}_+^2$ (since, as in the previous cases, from equation (5.6) we conclude that F_i is equivalent to Lebesgue measure in $\mathbb{R}_+$, $\forall i$), where $\forall k, l$,

$k \neq l$:

$$g_{k,l}(x_k) = \frac{R_{l,k}}{2R_{k,k}} a_{k,k} p_k(0^+) \frac{\frac{q_k(x_k)}{\theta_k + M_k - N_k} + \frac{r_k(x_k)}{\theta_k + M_k + N_k}}{q_k(x_k) + r_k(x_k)}$$

Thus, under all the previous requirements, if the stationary setting and its corresponding density p, satisfying conditions c3) and c4), exist, then:

$$p(x) = \prod_{i=1}^{n} \frac{p_i(0^+)}{2} \{q_i(x_i) + r_i(x_i)\}$$

for a.e. $x \in \mathbb{R}^n_+$.

6. Concluding Remarks

Although we have provided necessary and sufficient conditions for product-form densities to exist in the stationary regime, we did not discuss the conditions for the existence of this regime in the paper. However, we believe the conditions derived in Section 5 are the appropriate ones on each of the cases considered there, in the separable setting, of course. We also believe that product-form stationary densities are not possible under more general dependence of b and a on $x \in \mathbb{R}^n_+$ than the ones we have considered in this paper, or under a more general, non-separable, structure for the jumps. Moreover, in this paper we have considered the case of positive jumps. However, all the results in the paper still hold in the more general case of both, positive and negative jumps. For this, if the jump introduced by Π in the $i-th$ coordinate at time t is such that $X^i_t < 0$, then we assume that X^i_t is set to zero at that instant in order to be always non-negative. Therefore, Π can be assumed to be such that, $\forall t \in \mathbb{R}_+$, $\int_{t-}^{t} \int_{\mathbb{R}^n} z\Pi(ds, dz) \geq -X_{t-}$ (componentwise), $P-a.s.$, and then the regulator (or reflection map) process (Z_t) is still continuous. In addition, in this case condition c1) is of course extended to $\int_{\mathbb{R}^n_+ \times \mathbb{R}^n} |z_i| \Phi(x) k(x,z) dz dx = \int_{\mathbb{R}^n_+ \times \mathbb{R}} |z_i| \Phi(x) k_i(x, z_i) dz_i dx < +\infty$, $\forall i$. Note that models including negative jumps are important in risk theory or in financial models where claims arise at random times.

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