JOINT INVARIANT SIGNATURES FOR CURVE RECOGNITION

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ABSTRACT. We present a general method for recognizing curves modulo a Lie group action. This method is based on moving frames and consists in constructing a joint invariant signature which is robust and of minimal dimension. The example of planar curve recognition modulo equi-affine transformations is discussed in details.

1. INTRODUCTION

This paper is the sequel to a warm up paper on polygon recognition [3]. We are interested in curve recognition modulo Lie group actions, which is an important problem in computer vision. Very often, objects are represented by the boundary of their projection onto a plane (picture). Generally, the position and orientation of the curve on the picture are irrelevant informations for the purpose of identifying the object. Depending on how the picture is taken, other types of (Lie group) transformations represent irrelevant variations in the shape of the curve, e. g. equi-affine, affine, similarity and projective transformations. The problem of recognizing the object thus reduces to recognizing curves modulo a Lie group action.

One solution to this problem was proposed by Calabi et al. [5] and consists in constructing a signature parameterized by some well chosen differential invariants. The differential invariant signature of a curve is a representative for the whole equivalence class of this curve. In other words, two curves are in the same equivalence class if and only if their signature is the same. For planar curve recognition up to rotation and translation, the authors proposed the use of a signature curve parameterized by (κ, κ_s), where κ is the Euclidean curvature and κ_s its derivative with respect to arc-length. They also suggested a signature for curve recognition modulo equi-affine transformations. Although these differential invariant signatures are promising, one problem is their sensitivity to noise and round off errors. To improve the results obtained on discrete data, the use of numerically invariant approximations for these differential signature curves [5, 2] was suggested.

Yet another type of signature was proposed by Peter Olver in [10] as a solution to the problem of noise sensitivity. Instead of using differential invariants to parameterize a one-dimensional signature, the author proposed parameterizing a higher dimensional signature with a different type of invariants called *joint invariants*. Denote by $M^{\times(n)}$ the Cartesian product of n copies of a manifold $M^{\times(n)} = M \times \ldots \times M$ (n-times). The action of G on M can be prolonged unto $M^{\times(n)}$ by setting $g \cdot (z_1, \ldots, z_n) = (g \cdot z_1, \ldots, g \cdot z_n)$, for all $z_1, \ldots, z_n \in M$ and all $g \in G$. A joint invariant is an invariant of the prolonged action of G on $M^{\times(n)}$.

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More precisely, it is a real valued function $J : M^{\times(n)} \to \mathbb{R}$ which remains unchanged under the simultaneous action of G on the n points of M, in other words $J(g \cdot z_1, \ldots, g \cdot z_n) = J(z_1, \ldots, z_n)$, for all $z_1, \ldots, z_n \in M$ and all $g \in G$. In general, joint invariants are more noise resistant than differential invariants simply because they do not involve any derivative. The simplest joint invariant of the action of the Euclidean group on $\mathbb{R}^2$ is the distance between two points. In fact, any other Euclidean joint invariant is a function of distances. The simplest joint invariant of the action of the equi-affine Lie group on the plane is the signed area of the triangle spanned by three ordered points. As Élie Cartan has shown [6], the solution to many equivalence problems lies in the functional relationships between some well chosen invariants. In many cases, we can see this relationship simply by plotting the invariants. For example, by plotting the Euclidean distances between enough points on a curve, one is able to completely characterize the curve modulo rotations, translations and reflections. In fact, the six pairwise Euclidean distances defined by four points on the curve are enough for this purpose. A four-dimensional signature in $\mathbb{R}^6$ can thus be constructed by evaluating these six distances on all possible four points on the curve.

However, there is no a priori reason to believe that the signature of a curve should be more than one-dimensional. In fact, a one-dimensional signature is quite desirable as it insures a minimal complexity order for the recognition algorithm. Also, there is no apparent reason why the signature of a curve in an m-dimensional manifold M should be parameterized by more than m invariants. For example, since a planar curve is specified by two quantities, we see no reason why its signature should be specified by more than two quantities. Indeed, we can show that, if some slight regularity conditions hold, then one can obtain a certain set of m joint invariants using the moving frame method, and use them to parameterize a signature curve. For simplicity, we shall restrict ourselves to the case of a Lie group action on a two-dimensional smooth (Hausdorff) manifold M^2, although the theoretical results contained in [3] allow for a generalization to Lie group actions on any smooth (Hausdorff) manifold.

An interesting fact is that a signature parameterized by joint invariants no longer requires the curve to be differentiable, as opposed to the differential invariant signatures. This observation led us to consider using joint invariant signatures for recognizing curves which are not necessarily smooth (e. g. polygons). In fact, discontinuities in the derivatives provide possible choices for the distinguished points called landmarks, which are the key to constructing a simple signature. Some other (more robust) possibilities for landmarks will also be discussed. In a sense, we trade off high dimensionality for the use of landmarks. We believe that, with a good choice of landmarks, this will yield efficient and noise resistant curve recognition algorithms.

2. Curve Recognition up to Orientation and Area Preserving Affine Transformations

2.1. Curve Segment Recognition Modulo $SA(2)$. Let us start with a simple example of practical interest. Consider the action of the equi-affine group $SA(2)$ on $\{z \in \mathbb{R}^2\}$ given by

$$\bar{z} = Az + b,$$

where $A \in SL(2, \mathbb{R})$ and $b \in \mathbb{R}^2$.

Let C be a planar curve segment. Assume that this curve segment can be parameterized as $C = \{\alpha(t) \in \mathbb{R}^2 | 0 \leq t \leq 1\}$ with $\alpha(t)$ a continuous and C^1 map such that $\alpha(0) \neq \alpha(1)$ and $|\alpha'(t)| \neq 0$, for all $t \in [0,1]$. For simplicity, we exclude the cases where the curve segment self-intersects although these can be treated by a similar method. The parameterization is introduced solely to simplify the exposition; it is not relevant for we are concerned with curves obtained from images. In other words, we are interested in the graph of $\alpha(t)$.

We want to determine whether two given curve segments belong to the same orbit under the action of $SA(2)$. The case where C is a straight line is trivial: any two straight lines are equivalent modulo $SA(2)$. When C is not a straight line, we take the two end points $\alpha(0)$ and $\alpha(1)$ of C, and label them ζ_1 and ζ_2 respectively. These two points serve as landmarks. Observe that the order of the labeling is arbitrary, since it depends on our choice of parameterization.

For reasons to be given later, this case requires the use of a minimum of three landmarks. Denote by $\Delta(z_1, z_2, z_3) = \frac{1}{2}(z_3 - z_1) \times (z_2 - z_1)$ the signed area of the triangle with vertices z_1, z_2, z_3 and let $f : C \to \mathbb{R}$ be the function $f(z) = |\Delta(\zeta_1, \zeta_2, z)|$. If f reaches a local maximum in a unique point on C, then we let the third landmark ξ be this point. Observe that if two curves segments C and $\bar{C}$ on which f reaches a local maximum in a unique point are equivalent modulo $SA(2)$, then their landmarks ζ_1, ξ, ζ_2 and $\bar{\zeta}_1, \bar{\xi}, \bar{\zeta}_2$ respectively are also equivalent modulo $SA(2)$ (up to the choice of ordering of the end points). This is what we call *equivariance up to order reversion* of the landmarks. It provides us with an easy equivalence test since the equivalence class of three distinct non-collinear points in the plane is characterized by their signed area (a well know fact in affine geometry).

Lemma 2.1. *Let $C = \{\alpha(t)\}$ and $\bar{C} = \{\bar{\alpha}(t)\}$ be two curve segments on which f reaches a unique local maximum at ξ and $\bar{\xi}$ respectively. If C and $\bar{C}$ are equivalent under $SA(2)$, then*

$$\Delta(\alpha(0), \xi, \alpha(1)) = \pm\Delta(\bar{\alpha}(0), \bar{\xi}, \bar{\alpha}(1)).$$

In general, f may reached a local maximum in many points, including whole segments, on a curve. Since the property of *being a local maximum* of f is preserved under the action of $SA(2)$, local maxima can be used to simplify the search for equivalent curve segments. For example, if f reaches a local maximum only at a finite number of points $\xi_1, \xi_2, \ldots, \xi_k$ (labeled in increasing order according to the parameterization), then one can reject any other curve segment $\bar{C}$ for which the preimage of the local maxima of f ordered according to the parameterization of C neither belong to the equivalence class of $(\xi_1, \xi_2, \ldots, \xi_k)$ nor of $(\xi_k, \ldots, \xi_2, \xi_1)$. In particular, if this preimage is not finite of if it is finite but has a different cardinality, then it must be rejected. It is an easy task to test whether two strings of equal length belong to the same equivalence class, as we now explain.

Consider the two following four-point joint invariants of the action of $SA(2)$:

$$I_1(z_1, z_2, z_3, z_4) = \Delta(z_2, z_3, z_4) \quad \text{and} \quad I_2(z_1, z_2, z_3, z_4) = \Delta(z_1, z_2, z_4).$$

Define the map $S : (\mathbb{R}^2)^{\times(k)} \to \mathbb{R}^{k\times 2}$ by

$$S(z_1,\dots,z_k) = \begin{pmatrix} I_1(z_1,z_2,z_3,z_4), & I_2(z_1,z_2,z_3,z_4) \\ I_1(z_2,z_3,z_4,z_5), & I_2(z_2,z_3,z_4,z_5) \\ \vdots & \vdots \\ I_1(z_{k-2},z_{k-1},z_k,z_1), & I_2(z_{k-2},z_{k-1},z_k,z_1) \\ I_1(z_{k-1},z_k,z_1,z_2), & I_2(z_{k-1},z_k,z_1,z_2) \\ I_1(z_k,z_1,z_2,z_3), & I_2(z_k,z_1,z_2,z_3) \end{pmatrix}.$$

Lemma 2.2. *Let $k \geq 3$. If $(z_1,\dots,z_k),(\bar{z}_1,\dots,\bar{z}_k) \in (\mathbb{R}^2)^{\times(k)}$ are two strings such that no three consecutive of their vertices lie on a straight line (e. g. all consecutive points are distinct), then there exists $g \in SA(2)$ such that $g\cdot(z_1,\dots,z_k) = (\bar{z}_1,\dots,\bar{z}_k)$ if and only if $S(z_1,\dots,z_k) = S(\bar{z}_1,\dots,\bar{z}_k)$.*

The proof is a particular case of the proof of Theorem 2.5. As a corollary, we have the following equivalence test.

Corollary 2.3. *Let $k \geq 3$ and assume that f reaches a local maximum at the points $\{\xi_1,\xi_2,\dots,\xi_k\}$ and $\{\bar{\xi}_1,\bar{\xi}_2,\dots,\bar{\xi}_k\}$ on C and $\bar{C}$ respectively. Assume that the ξ_i's and $\bar{\xi}_i$'s are labeled in increasing order with respect to the parameterization and consider the strings $(\xi_1,\xi_2,\dots,\xi_k)$ and $(\bar{\xi}_1,\bar{\xi}_2,\dots,\bar{\xi}_k)$.*
If [2] *$\Delta(\xi_i,\xi_{i+1},\xi_{i+2}),\Delta(\bar{\xi}_i,\bar{\xi}_{i+1},\bar{\xi}_{i+2}),\neq 0$ for all $i = 1,\dots,k-2$, then $C \equiv \bar{C}$* mod *$SA(2)$ implies that either $S(\xi_1,\xi_2,\dots,\xi_k) = S(\bar{\xi}_1,\bar{\xi}_2,\dots,\bar{\xi}_k)$ or*
$S(\xi_k,\dots\xi_2,\xi_1) = S(\bar{\xi}_1,\bar{\xi}_2,\dots,\bar{\xi}_k)$.

One can think of many possible variations of the previous equivalence test. For example, we could have considered the string $(\zeta_1,\xi_1,\dots,\xi_k,\zeta_2) \in (\mathbb{R}^2)^{\times(k)}$ of length $(k+2)$ defined by the points where f reaches a maximum $\xi_1,\dots,\xi_k$ together with the end points ζ_1 and ζ_2. By Lemma 2.2, the equivalence of such strings of landmarks can be checked by comparing $S(\zeta_1,\xi_1,\dots,\xi_k,\zeta_2)$ and $S(\bar{\zeta}_1,\bar{\xi}_1,\bar{\xi}_2,\dots,\bar{\xi}_k,\bar{\zeta}_2)$. In the case where f reaches a local maximum on a whole segment of the curve C, then the end points of this interval can also be used in a test string to check for the equivalence of curve segments. Using a longer string of landmarks provides, of course, a better equivalence test, one that will be more likely to rule out non-equivalence candidates.

If no string equivalence test rules out equivalence, then we need to use all the relevant information contained in the curve segment. Let ξ_+ be the first point at which f reaches a local maximum (according to our choice of parameterization) and let ξ_- be the last such point. Let $\Sigma_+(C)$ be the curve parameterized by $(I_1(\zeta_1,\xi_+,\zeta_2,\alpha(t)), I_2(\zeta_1,\xi_+,\zeta_2,\alpha(t)))$, for $0 \leq t \leq 1$, and let $\Sigma_-(C)$ be the curve parameterized by $(I_1(\zeta_2,\xi_-,\zeta_1,\alpha(1-t)), I_2(\zeta_2,\xi_-,\zeta_1,\alpha(1-t)))$, for $0 \leq t \leq 1$.

Definition 2.4. Given a choice of parameterization for a curve segment $C = \{\alpha(t)\}$, we define its signature $\Sigma(C)$ as the ordered pair of curves

$$\Sigma(C) = (\Sigma_+(C),\Sigma_-(C)).$$

Theorem 2.5. *Two curve segments $C = \{\alpha(t)\}$ and $\bar{C} = \{\bar{\alpha}(t)\}$ on which f reaches a unique local maximum are equivalent if and only if either $(\Sigma_+(C),\Sigma_-(C)) = (\Sigma_+(\bar{C}),\Sigma_-(\bar{C}))$ or $(\Sigma_-(C),\Sigma_+(C)) = (\Sigma_+(\bar{C}),\Sigma_-(\bar{C}))$.*

[2] Note: If three consecutive ξ_i's lie on a straight line, one can try to permute the points to avoid collinearity and perform the equivalence test on the permuted points.

Proof. Observe that the signature is a function solely of signed areas, which are invariants of the equi-affine group action on the plane. Moreover, except for the choice of orientation, our procedure to determine the landmarks will consistently lead to the same three points modulo $SA(2)$. So the signature of two equivalent curve segments will be the same up to a permutation of its two curve components.

Now, suppose we are given two curve segments C and $\bar{C}$ with the same signature up to a permutation of the two curve components. We can assume that $(\Sigma_+(C), \Sigma_-(C)) = \big(\Sigma_+(\bar{C}), \Sigma_-(\bar{C})\big)$ by reparameterizing one of the curve with $1-t$ instead of t, thus reversing the direction of travel, if necessary. We then have $\Sigma_+(C) = \Sigma_+(\bar{C})$. Consider the end points of these curve segments. One of them lies on the y axis and the other lies on the x axis. Observe that ζ_1, ζ_2 and ξ_+ must be distinct, therefore none of the end points of the signature actually lies at the origin. The end point lying on the y axis corresponds to the first points ζ_1 and $\bar{\zeta}_1$ of each curve segment. By looking at the second component of this end point, we know the value of $\Delta(\zeta_1, \xi_+, \zeta_2)$ and $\Delta(\bar{\zeta}_1, \bar{\xi}_+, \bar{\zeta}_2)$, which must be equal by assumption. It is a well known fact in affine geometry that three distinct non-collinear points in $\mathbb{R}^2$ can be mapped unto each other using an equi-affine transformation if and only if their signed area is the same. So there exists $g \in SA(2)$ such that $g \cdot (\zeta_1, \xi_+, \zeta_2) = (\bar{\zeta}_1, \bar{\xi}_+, \bar{\zeta}_2)$.

Observe that the invariants I_1 and I_2 are such that given three distinct points z_1, z_2 and z_3, the value of $I_1(z_1, z_2, z_3, z_4)$ and $I_2(z_1, z_2, z_3, z_4)$ uniquely determines z_4, provided that $I_1(z_1, z_2, z_3, z_4) \neq 0$. This means that all points $z \in C$ such that $\Delta(\xi_+, \zeta_2, z) \neq 0$ are mapped to corresponding points $\bar{z} \in \bar{C}$ by g. By continuity of the curve segments and since the parameterization has nonzero speed, we conclude that $g \cdot C = \bar{C}$. □

Note that, although only three landmarks are used in our method, one could certainly construct a signature that uses more than three landmarks, as we shall do for the case of closed curves.

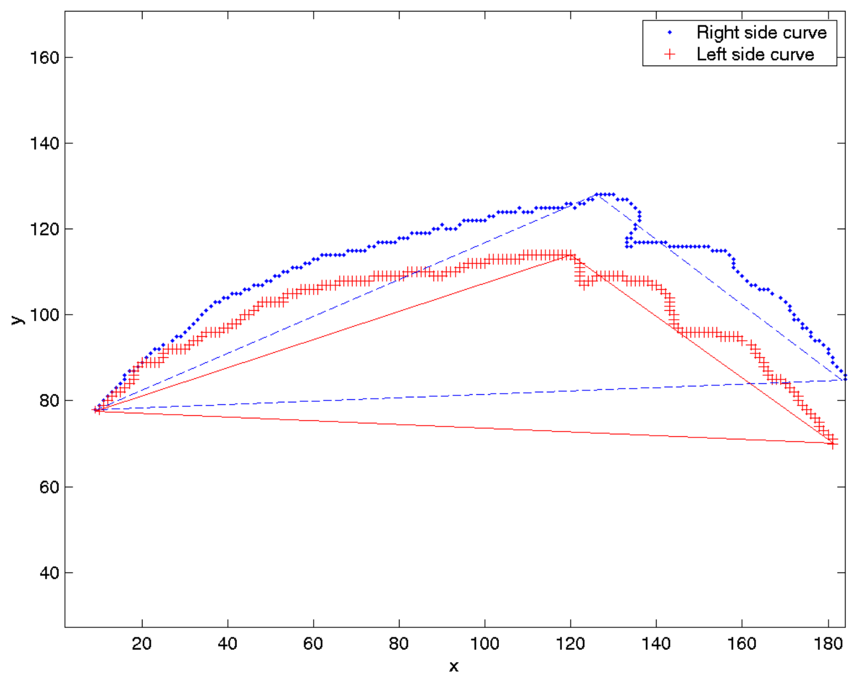
We took the picture of a leaf and segmented it (see Figure 1) in order to test the noise resistance of our method. We considered the boundaries of the left and right sides of the leaf as two curve segments. The left side curve was flipped before being compared to the right side curve. Note that no preprocessing was done on either curves. The end points were used as landmarks and, for each curve, a third landmark was found by taking the point on the boundary spanning the triangle of maximal area, $A_{max}(\text{right})$ and $A_{max}(\text{left})$ respectively. These three landmarks are represented as the vertices of a triangle on Figure 2. The signature of both segments is displayed in Figure 3 together with a third signature obtained by multiplying the signature of the left side of the leaf by $\frac{A_{max}(\text{right})}{A_{max}(\text{left})}$. Despite significant local variations in the shapes of the sides, the signature of the right side curve and the rescaled signature of the left side curve are surprisingly similar.

2.2. **Closed Curve Recognition Modulo $SA(2)$.** Let C be a closed, non-self-intersecting planar curve parameterized by $\{\alpha(t) \in \mathbb{R}^2 \,|\, 0 \leq t \leq 1\}$ with α continuous and piecewise differentiable, $\alpha(0) = \alpha(1)$ and $\alpha'(t) \neq 0$ on the differentiable pieces.

The first thing to do is to choose a minimum of three landmarks. Again, there are many ways to do this. For example, consider the affine curvature κ at every point of the curve, which in local coordinates $(x, y) \in \mathbb{R}^2$ with $y = u(x)$ is given by $\kappa(x) = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9u_{xx}^{8/3}}$. Observe that κ is not defined at inflection points of

FIGURE 1. Segmented Leaf

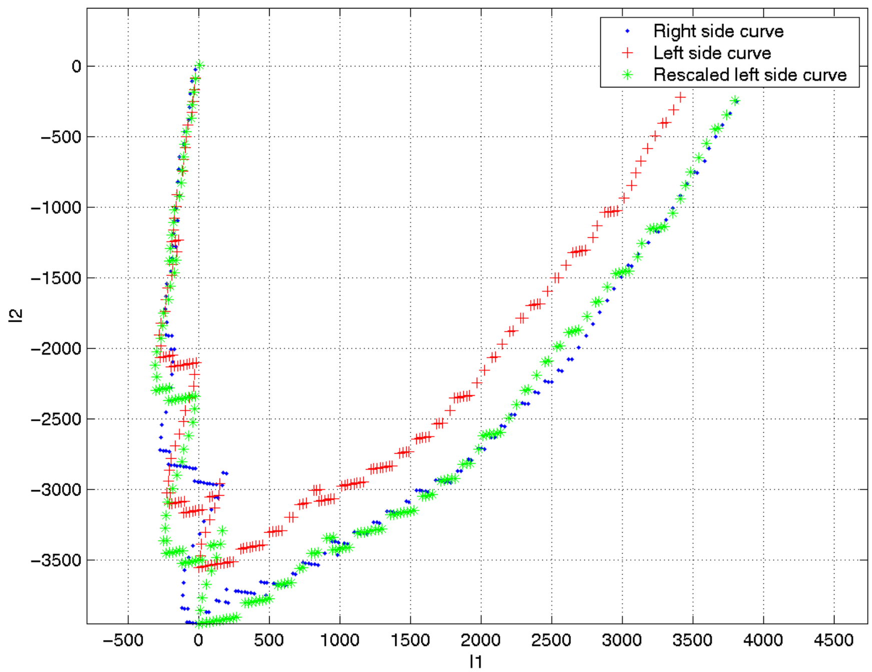
FIGURE 2. The Two Side Curves of the Leaf



the curve. Define the map $g : C \to \mathbb{R}$ by $g(z) = |\kappa(z)|$. One possibility for the landmarks is to take all the points on C where the function g either is not defined or reaches a local maximum.

For a closed (regular) and C^4 curve, one can show that there exist at least three points on the curve where the affine curvature reaches a local maximum [8].

FIGURE 3. Equi-affine Invariant Signatures of the Two Side Curves of the Leaf



The case where the affine curvature is constant corresponds to ellipses and can be treated separately. Indeed two ellipses can be mapped to each other by an affine transformation if and only if their area is the same. If g reaches a local max on a C^4 curve at a finite number of points $\{\zeta_1, \ldots, \zeta_k\}$, $k \geq 3$, then these points can be used as landmarks. One can also include the endpoints of the intervals on which g reaches a local max, if such an interval exists.

If the curve is merely piecewise C^4, then one can include the points where the affine curvature doesn't exist or is not continuous in the set of landmarks. If doing so yields less than three landmarks, then one can divide up the differentiable pieces of the curve into intervals of equal affine arc-length and use the boundary points of these intervals as landmarks. Adding the points where g reaches a local maximum is also a possibility.

A different approach to defining landmarks would be to use the affine skeleton [1] of the curve C. The affine skeleton being an equivariant structure, we could, for example, define landmarks directly on the skeleton and use these as landmarks for the curve. One could think of using end points or junction points of the skeleton, which are characteristics that are preserved under the action of $SA(2)$. In any event, the essential is to obtain a string of at least three (robust) landmarks and the technique used to achieve this goal matters little.

Let $C = \{\alpha(t)\}$ be a closed curve with k landmarks $(\zeta_1, \ldots, \zeta_k) \in (\mathbb{R}^2)^{\times(k)}$. Let $\mathbb{Z}_k$ act on $(\mathbb{R}^2)^{\times(k)}$ by cyclically permuting the order of the k landmarks. Let $\mathbb{Z}_2$ act on $(\mathbb{R}^2)^{\times(k)}$ by reversing the order of the k landmarks. Consider $\mathbb{H}_k =< \mathbb{Z}_k, \mathbb{Z}_2 >$, the group of transformations generated by the action of $\mathbb{Z}_k$ and $\mathbb{Z}_2$ on $(\mathbb{R}^2)^{\times(k)}$, and the point $\langle \zeta_1, \ldots, \zeta_k \rangle \in (\mathbb{R}^2)^{\times(k)} \mod \mathbb{H}_k$. We identify this point with a polygon $P = P(C)$ in $\mathbb{R}^2$. To simplify the exposition, we set $\zeta_{k+1} = \zeta_1$, $\zeta_{k+2} = \zeta_2$, and so on. By construction, the polygon $P(C)$ associated to a curve C is equivariant, in the sense that if $\bar{C} = g \cdot C$, then $P(\bar{C}) = g \cdot P(C)$. The equivalence of two polygons can be checked in $O(k)$ steps using the map S defined in subsection 2.1. In fact, we have the following simple equivalence test, which can be proved by the same arguments as the ones used in the proof of 2.5.

Theorem 2.6. [3] *Let $k \geq 3$ and let $P = \langle p_1, \ldots, p_k \rangle$ and $\bar{P} = \langle \bar{p}_1, \ldots, \bar{p}_k \rangle$ be two polygons with no three consecutive points lying on a straight line. Then $P \equiv \bar{P} \mod SA(2)$ if and only if*

$$\begin{aligned} S(p_1, p_2, \ldots, p_k) &= S(\bar{p}_1, \ldots, \bar{p}_k) \mod \mathbb{Z}_k \\ \textit{or } S(p_k, \ldots, p_2, p_1) &= S(\bar{p}_1, \ldots, \bar{p}_k) \mod \mathbb{Z}_k. \end{aligned}$$

If comparison of the polygons doesn't rule out equivalence, then we are ready to perform the ultimate equivalence test. For $i = 1, \ldots, k$, let $\Sigma_{i+}(C)$ be the planar curve segment parameterized by $I_1(\zeta_i, \zeta_{i+1}, \zeta_{i+2}, z)$ and $I_2(\zeta_i, \zeta_{i+1}, \zeta_{i+2}, z)$, for $z \in C$ going from ζ_{i+3} to ζ_{i+2}. Consider $\Sigma_+(C)$, the string of k curves

$$\Sigma_+(C) = (\Sigma_{1+}, \Sigma_{2+}, \ldots, \Sigma_{k+}) .$$

Observe that the signature for polygons used in Theorem 2.6 corresponds to the first point of each curve segment Σ_{i+}. Similarly, let Σ_{i-} be the planar curve segment parameterized by $I_1(\zeta_i, \zeta_{i-1}, \zeta_{i-2}, z)$ and $I_2(\zeta_i, \zeta_{i-1}, \zeta_{i-2}, z)$ for $z \in C$ going from ζ_{i-3} to ζ_{i-2}. We define the signature $\Sigma(C)$ of a closed curve $C = \{\alpha(t)\}$ as the pair of curves $\Sigma(C) = (\Sigma_+(C), \Sigma_-(C))$. As a corollary of 2.5, we have the following theorem.

Theorem 2.7. *Two closed simple curves with a string of $k \geq 3$ landmarks are equivalent under $SA(2)$ if and only if either $\Sigma_+(C) \equiv \Sigma_+(\bar{C}) \mod \mathbb{Z}_k$ or $\Sigma_-(C) \equiv \Sigma_+(\bar{C}) \mod \mathbb{Z}_k$.*

3. Generalization to Other Lie Groups

3.1. **Theoretical Foundations.** Let G be a δ-dimensional Lie group acting on an m-dimensional smooth (Hausdorff) manifold M. In this paper, we assume that $M = M^2$ is a two-dimensional manifold, although the theory developed in [3] applies to manifolds of any dimension. Our recipe to build a signature curve uses two ingredients: 1) a set of two suitable invariants, 2) a set of distinguished points called landmarks. Let us first talk about the landmarks.

The only property that we require of the landmarks is that they be equivalent on equivalent curves. This is what we call *equivariance of the landmarks* which we now define precisely. Let $\mathcal{L}$ be a rule that associates to every curve C a finite string of points $\mathcal{L}(C) = (\xi_1, \ldots, \xi_k) \in (M^2)^{\times(k)}$.

Definition 3.1. We say that $\mathcal{L}$ is *equivariant* if

$$\mathcal{L}(g \cdot C) = g \cdot \mathcal{L}(C), \text{ for all } g \in G.$$

We say that $\mathcal{L}$ is *equivariant up to H_k*, for some subgroup of permutations $H_k \subset S_k$, if for any $\bar{C} = g \cdot C$ for some $g \in G$, there exists $h \in H_k$ such that

$$\mathcal{L}(\bar{C}) = g \cdot h \cdot \mathcal{L}(C).$$

Definition 3.2. If $\mathcal{L}$ is an equivariant rule, C a curve segment and if $\mathcal{L}(C) = (\xi_1, \ldots, \xi_k)$, then we say that the points $\{\xi_1, \ldots, \xi_k\}$ are (equivariant) landmarks for C.

The freedom in the choice of landmarks comes from the infinitely many possible choices of equivariant rules $\mathcal{L}$. In practical applications, the choice of $\mathcal{L}$ should be made in such a way to insure robustness of the landmarks. For example, for the third landmark on planar curves under the action of $SA(2)$, we could have decided to take the first (or last) global maximum of the function f, but for noisy data representing curves with more than one local max, this would not yield robust landmarks. We could also have taken the first point encountered where the affine invariant curvature reaches a maximum or doesn't exist, but in general, the more derivatives the rule $\mathcal{L}$ relies on, the more sensitive the results. So a perhaps better possibility is to pick the point that lies at half the total affine arc-length of the curve segment. There are many other possibilities for choosing landmarks and performance of any one type of landmarks will always depend on the type of curve segments under consideration.

The next ingredient on our list are the invariants that will parameterize the signature. We impose the condition that only two invariants should be used, so as to not increase the complexity of the recognition algorithm unnecessarily.

Definition 3.3. We say that the group action is *semi-regular* if all the orbits have the same dimension. If, in addition, for all $z \in M$, there exists a collection of arbitrarily small neighborhoods whose intersection with the orbit through z is connected, then we say that the action is *regular*.

Most of our results are based on the following theorem due to Frobenius.

Theorem 3.4. [7] *If G acts on an open set $O \subset M$ semi-regularly with s dimensional orbits, then $\forall x_0 \in O$ there exist $m - s$ functionally independent local invariants $I_1, \ldots, I_{m-s}$ defined on a neighborhood U of x_0 such that any other local invariant I defined near x_0 is a function $I = f(I_1, \ldots, I_{m-s})$. If the action of G is regular, then the local invariants can be taken to be invariants in a neighborhood of x_0, and two points $x_1, x_2 \in U$ are in the same orbit if and only if $I_i(x_1) = I_i(x_2)$, $\forall i = 1, \ldots, m - s$.*

The set $\{I_1, \ldots, I_{m-s}\}$ is often called a *complete fundamental set of invariants.* The moving frame normalization method, as explained in detail in [9] Chapter 8, is an algorithm to construct a complete fundamental set if invariants. The idea is to build a right-equivariant map $\rho : M \to G$ called *moving frame.* By right-equivariance, we mean that $\rho(g \cdot x) = \rho(x)g^{-1}$, for all $g \in G$ and all $x \in M$. A moving frame can be constructed following a step-by-step method. Given local coordinates $x = (x_1, \ldots, x_m)$ in a neighborhood of $x_0 \in M$, we consider the m coordinates of the vector $\rho(x) \cdot x$, which contain a complete fundamental set of invariants define in a neighborhood of x_0.

In the following, we shall make use of this interesting property of the maximal orbit dimension.

Lemma 3.5. [4] *Let s_n denote the maximal orbit dimension of the prolonged action of G on $M^{\times(n)}$. If $s_n = s_{n+1}$, then $s_{n+j} = s_n$, for all $j \in \mathbb{N}$.*

Definition 3.6. We say that a group action of G on a manifold M is *effective* if the subgroup $\{g \in G | g \cdot z = z, \forall z \in M\} = \{e\}$ is trivial. We say that a group action of G on a manifold M is *effective on subsets* if for all open subset $U \subset M$ the subgroup $\{g \in G | g \cdot z = z, \forall z \in U\} = \{e\}$ is trivial.

Our method also relies strongly on the following important fact.

Theorem 3.7. [4] *There exists a minimal n_0 such that G acts with δ-dimensional orbits on a dense and open subset $\mathcal{D}$ of $M^{\times(n_0)}$ if and only if G acts (locally) effectively on subsets of M.*

A general method for constructing suitable invariants to characterize polygons in an m-dimensional manifold modulo a Lie group action is explained in details in [3]. It turns out that the same invariants are suitable for characterizing curves. We shall summarize these results here for the particular cases of Lie group actions on a two-dimensional manifold M^2.

We are assuming for the rest of this paper that G acts effectively on subsets of M^2 so that n_0 exists. Observe that the action of G on the (open) set of points belonging to orbits of maximal dimensions is (at least) semi-regular. So Theorem 3.4 guarantees the existence of fundamental invariants in a neighborhood of any such points, and we can construct these invariants using the moving frame normalization method.

Let n^* be the minimum integer n such that the maximal orbit dimension s_n of the action of prolonged action of G on $(M^2)^{\times(n)}$ is strictly smaller than $2n$, the dimension of $(M^2)^{\times(n)}$. In other words, n^* is the minimal number of points on which non-trivial joint invariants can depend (by Theorem 3.4). Observe that $n_0 \geq n^* - 1$. By considering sets of fundamental invariants on open subsets of $(M^2)^{\times(n^*)}$, $(M^2)^{\times(n^*+1)}$, ... , $(M^2)^{\times(n_0)}$, $(M^2)^{\times(n_0+1)}$ successively, we can cook up a set of two invariants I_1 and I_2 that are suitable for parameterizing a signature curve. Recall that δ denotes the dimension of G.

Lemma 3.8. *Let s_n denote the orbit dimension of the prolonged Lie group action on $(M^2)^{\times(n)}$. Then the following relations hold:*

$$\begin{aligned} s_n &= 2n, \textit{for all } n < n^*, \\ s_n &= s_{n-1} + 1, \textit{for all } n_0 \geq n \geq n^*, \\ s_n &= s_{n-1}, \textit{for all } n > n_0, \\ \textit{(and therefore) } n_0 &= \delta + 1 - n^*. \end{aligned}$$

Proof. By definition of n^*, we have $s_n = 2n$, for all $n > n^*$ and $s_{n^*} < 2n^*$. By Lemma 3.5, $s_n - s_{n-1} = 1$ or 2, whenever $n \leq n_0$, while $s_n - s_{n-1} = 0$, whenever $n > n_0$. However, it turns out that $s_n - s_{n-1} < 2$ for any $n \geq n^*$. Here is why.

- For $n = n^*$. If $s_{n^*} - s_{n^*-1} = 2$, then $s_{n^*} = 2n^*$ which contradicts the definition of n^*.
- For all $n > n^*$. The number of n-point joint invariant $\#_n$ is $\#_n - \#_{n-1} \geq 1$, since if $I(x_1, \ldots, x_{n^*})$ is an n^*-point joint invariant, then $\bar{I}(x_1, \ldots, x_n) := I(x_{n-n^*+1}, \ldots, x_n)$ can be used as a fundamental n-point joint invariant.

Therefore we have $s_{n^*-1} = 2(n^* - 1)$, $s_n - s_{n-1} = 1$, for all $n_0 \geq n \geq n^*$ and finally $s_n - s_{n-1} = 0$, for all $n > n_0$. We conclude that $\delta = 2(n^* - 1) + (n_0 - n^* + 1) = n^* + n_0 - 1$. □

3.2. **The algorithm.** For simplicity, we divide the exposition of our algorithm into two distinct cases.

Case 1: $n_0 = n^* - 1$. Observe that, by Lemma 3.8, this case requires the dimension of the group δ to be even. It includes the similarity Lie group $SIM(2)$ generated by dilations, rotations and translations in the plane, which is discussed in [3]. The distinguishing property of such group actions is that there are exactly two fundamental invariants defined in some neighborhood U of any point $x_0 \in (M^2)^{\times(n^*)}$ belonging to an orbit of maximal dimension $s_{n^*} = \delta$, as proved by Theorem 3.4 and Lemma 3.8. These invariants can be obtained using the moving frame normalization method.

Lemma 3.9. *Assume $n_0 = n^* - 1$. Let $x_0 \in (M^2)^{\times(n_0+1)}$ be a point belonging to an orbit of maximal dimension $s_{n_0+1} = \delta$ and let $J_1, J_2 : U \subset (M^2)^{\times(n_0+1)} \to \mathbb{R}$ be a complete fundamental set of invariants defined in a neighborhood of x_0. There exists a neighborhood $\bar{U} \subset U$ of x_0 such that on $\{(z_1, \ldots, z_{n_0+1}) \in \bar{U}\}$, the last point z_{n_0+1} can be expressed as a function*

$$z_{n_0+1} = f\left(z_1, \ldots, z_{n_0}, J_1(z_1, \ldots, z_{n_0+1}), J_2(z_1, \ldots, z_{n_0+1})\right).$$

Proof. Consider the 2-by-2(n_0+1) Jacobian matrix

$$\frac{\partial(J_1, J_2)}{\partial(z_1, \ldots, z_{n_0+1})}$$

which, by functional independence of J_1 and J_2, has rank two on an open and dense subset of U. Since there are no n_0-point joint invariants, the 2-by-2 sub-matrix

$$\frac{\partial(J_1, J_2)}{\partial z_{n_0+1}}$$

must also have rank two on an open and dense subset of U and the conclusion follows by the implicit function theorem. □

Assuming that g acts transitively on the restriction

$$\Pi_{1,\ldots,n_0}\bar{U} = \{(z_1, \ldots, z_{n_0}) \in (M^2)^{\times(n_0)} | \exists z_{n_0+1} \text{ s.t. } (z_1, \ldots, z_{n_0+1}) \in \bar{U}\},$$

then we can simply use $I_1 = J_1$ and $I_2 = J_2$ to parameterize a signature for curve segments.

Given a curve segment $C = \{\alpha(t)\}$ with $n_0 = n^* - 1$ landmarks $(\zeta_1, \ldots, \zeta_{n_0})$ in the direction of the parameterization and the same number of landmarks $(\lambda_1, \ldots, \lambda_{n_0})$ in the other direction, we write these landmarks as $[(\zeta_1, \ldots, \zeta_{n_0}), (\lambda_1, \ldots, \lambda_{n_0})]$. We define the joint invariant signature of $\Sigma(C)$ of the curve segment C as the pair of planar curves

$$\Sigma(C) = (\Sigma_+(C), \Sigma_-(C))$$

where $\Sigma_+(C)$ is the curve parameterized by $I_1(\zeta_1, \ldots, \zeta_{n_0}, \alpha(t))$ and $I_2(\zeta_1, \ldots, \zeta_{n_0}, \alpha(1-t))$, while $\Sigma_-(C)$ is the curve parameterized by $I_1(\lambda_1, \ldots, \lambda_{n_0}, \alpha(1-t))$ and $I_2(\lambda_1, \ldots, \lambda_{n_0}, \alpha(1-t))$, for $0 \le t \le 1$. Let $\mathbb{Z}_2$ act on the signature $\Sigma(C)$ by permuting its two curve components.

Theorem 3.10 (For curve segment recognition). *Let $\bar{U} \subset$ be an open set as described in Lemma 3.9. Assume that G acts transitively on the restriction $\Pi_{1,\ldots,n_0}\bar{U}$. Consider two curve segments C and $\bar{C}$ with landmarks $[(\zeta_1, \ldots, \zeta_{n_0}), (\lambda_1, \ldots, \lambda_{n_0})]$ and $\left[(\bar{\zeta}_1, \ldots, \bar{\zeta}_{n_0}), (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n_0})\right]$ respectively. If, for all $0 \le t \le 1$, we have $(\zeta_1, \ldots, \zeta_{n_0}, \alpha(t)), \left(\bar{\zeta}_1, \ldots, \bar{\zeta}_{n_0}, \bar{\alpha}(t)\right) \in \bar{U}$, as well as $(\lambda_1, \ldots, \lambda_{n_0}, \alpha(1-t)), \left(\bar{\lambda}_1, \ldots, \bar{\lambda}_{n_0}, \bar{\alpha}(1-t)\right) \in \bar{U}$. Then $C \equiv \bar{C} \mod G$ if and only if $\Sigma(C) \equiv \Sigma(\bar{C}) \mod \mathbb{Z}_2$.*

Proof. Invariance of the signature follows from the fact that it is parameterized by invariants and that the landmarks are equivariant up to a permutation of the ζ and λ components corresponding to reversing the direction of the parameterization.

To prove that the signature of C completely characterizes the equivalence class of C, observe that, by transitivity of the action of G on Ω, there always exists $g \in G$ such that $g \cdot (\zeta_1, \dots, \zeta_{n_0}) = (\bar{\zeta}_1, \dots, \bar{\zeta}_{n_0})$. Assuming that $\Sigma_+(C) = \Sigma_+(\bar{C})$ (reparameterize $\bar{C}$ if necessary), then by Lemma 3.9, $g \cdot C = \bar{C}$. □

We can also use $I_1 = J_1$ and $I_2 = J_2$ to parameterize a joint-invariant signature $\Sigma(C)$ for a closed curve C with $k \geq n_0 = n^* - 1$ landmarks $(\zeta_1, \dots, \zeta_k)$. Setting $(\zeta_{k+1}, \dots, \zeta_{2k}) = (\zeta_1, \dots, \zeta_k)$, we let $\Sigma(C) = (\Sigma_+(C), \Sigma_-(C))$ where

$$\Sigma_+(C) = (\Sigma_{1+}(C), \dots, \Sigma_{k+}(C))$$

is a string of k curves $\Sigma_{i+}(C)$ parameterized by $I_1(\zeta_i, \dots, \zeta_{n_0+i-1}, z)$ and $I_2(\zeta_i, \dots, \zeta_{n_0+i-1}, z)$ with $z \in C$ going from ζ_{n_0+i} to ζ_{n_0+i-1} and $\Sigma_-(C) = \Sigma_+(\{\alpha(1-t)\})$. As a corollary of 3.10, we have the following theorem.

Theorem 3.11 (For closed curve recognition). *Let $\bar{U} \subset$ be an open set as described in Lemma 3.9. Assume that G acts transitively on the restriction $\Pi_{1,\dots,n_0}\bar{U}$. Consider two closed simple curves C and $\bar{C}$ with $k \geq n_0$ landmarks $(\zeta_1, \dots, \zeta_k)$ and $(\bar{\zeta}_1, \dots, \bar{\zeta}_k)$ respectively. Setting $(\zeta_{k+1}, \dots, \zeta_{2k}) = (\zeta_1, \dots, \zeta_k)$ and $(\bar{\zeta}_{k+1}, \dots, \bar{\zeta}_{2k}) = (\bar{\zeta}_1, \dots, \bar{\zeta}_k)$, assume that, for all $i = 1, \dots, k$ and for all $z \in C$ between ζ_{n_0+i} and ζ_{n_0+i-1} we have $(\zeta_i, \dots, \zeta_{i+n_0-1}, z) \in \bar{U}$. Then $C \equiv \bar{C} \mod G$ if and only if either $\Sigma_+(C) = \Sigma_+(\bar{C}) \mod \mathbb{Z}_k$ or $\Sigma_-(C) = \Sigma_+(\bar{C}) \mod \mathbb{Z}_k$.*

Case 2: $n_0 > n^* - 1$. In addition to the equi-affine group action on the plane discussed previously, this case includes the action of the (full/special) Euclidean group acting on the plane and the action of $SL(2)$ on the Poincaré half-plane which were discussed in [3]. It is characterized by the fact that there exists a single fundamental invariant J defined in some neighborhood of any point $x_0 \in (M^2)^{\times(n^*)}$ belonging to an orbit of maximal dimension $s_{n^*} \leq \delta$. Again, this invariant can be obtained following the moving frame normalization method.

Lemma 3.12. *Assuming that $n_0 > n^* - 1$, let $(z_1, \dots, z_{n_0})$ be local coordinates for $(M^2)^{\times(n_0)}$ and let $\{J\}$ be a complete fundamental set of invariants defined in a neighborhood of a point $x_0 \in (M^2)^{\times(n_0)}$ belonging to an orbit of maximal dimension. There exists an open set $U_{n_0} \subset (M^2)^{\times(n_0)}$ such that*

$$\{J(z_i, \dots, z_{n^*+i-1})\}_{i=1}^{n_0-n^*+1}$$

is a complete fundamental set of invariants on U_{n_0}.

Proof. Let $J_i = J(z_i, \dots, z_{n^*+i-1})$, for $i = 1, \dots, n_0 - n^* + 1$. By Lemma 3.8, the maximal orbit dimension $s_{n_0} = \delta = n_0 + n^* - 1$. So, by Theorem 3.4, there are $2n_0 - s_{n_0} = 2n_0 - (n_0 + n^* - 1) = n_0 - n^* + 1$ fundamental invariants in a neighborhood of x_0. The conclusion follows by observing that the Jacobian matrix

$$\frac{\partial J_1, \dots, J_{n_0-n^*+1}}{\partial(z_1, \dots, z_{n_0})}.$$

has rank $n_0 - n^* + 1$ on an open and dense subset of an open subset U_{n_0} of $(M^2)^{\times(n_0)}$. □

Lemma 3.13. *Assuming that $n_0 \geq n^*$, let $\{J\}$ be a complete fundamental set of invariants defined in a neighborhood of a point $x_0 \in (\mathbb{R}^2)^{\times(n_0)}$ belonging to an orbit of maximal dimension. There exists an open set $U_{n_0+1} \subset (\mathbb{R}^2)^{\times(n_0+1)}$ and an invariant $H : U_{n_0+1} \to \mathbb{R}$ such that*

$$\{J(z_i, \ldots, z_{n^*+i-1})\}_{i=1}^{n_0-n^*+2} \cup \{H(z_1, \ldots, z_{n_0+1})\}$$

is a complete set of fundamental invariants on U_{n_0+1}, while

$$\{J(z_i, \ldots, z_{n^*+i-1})\}_{i=1}^{n_0-n^*+1}$$

is a complete fundamental set of invariants on the restriction

$$\Pi_{1,\ldots,n_0} U_{n_0+1} = \{(z_1, \ldots, z_{n_0}) \in (M^2)^{\times(n_0)} | \exists z_{n_0+1} s.t. (z_1, \ldots, z_{n_0+1}) \in U_{n_0+1}\}.$$

Moreover, we can choose U_{n_0+1} such that, on U_{n_0+1}, the last point z_{n_0+1} is a function

$$z_{n_0+1} = f\left(z_1, \ldots, z_{n_0}, J(z_{n_0-n^*+2}, \ldots, z_{n_0+1}), H(z_1, \ldots, z_{n_0+1})\right).$$

Proof. By Lemma 3.12, there exists $U_{n_0} \subset (M^2)^{\times(n_0)}$ on which $\{J(z_i, \ldots, z_{n^*+i-1})\}_{i=1}^{n_0-n^*+1}$ is a complete fundamental set of invariants. Let $J_i(z_1, \ldots, z_{n_0+1}) = J(z_i, \ldots, z_{n^*+i-1})$. Observe that there exists an open subset of $(M^2)^{\times(n_0+1)}$ on which the rank of the Jacobian matrix

$$\frac{\partial(J_1, \ldots, J_{n_0-n^*+2})}{\partial(z_1, \ldots, z_{n_0+1})}$$

is equal to $n_0 - n^* + 2$. This open subset can be taken as $\bar{U}_{n_0} \times U_1$, where $\bar{U}_{n_0} \subset U_{n_0}$ and $U_1 \subset M^2$. The number of functionally independent invariants on $(\mathbb{R}^2)^{\times(n_0+1)}$ is

$$\begin{aligned} 2(n_0+1) - s_{n_0+1} &= 2(n_0+1) - \delta, \\ &= 2(n_0+1) - (n_0 + n^* - 1), \text{ by Lemma 3.8}, \\ &= n_0 - n^* + 3, \end{aligned}$$

so there exists another invariant H and an open subset $U_{n_0+1} \subset \bar{U}_{n_0} \times U_1$ such that $\{J_1, \ldots, J_{n_0-n^*+2}, H\}$ is a complete fundamental set of invariants on U_{n_0+1}.

To show the second part of the statement, observe that the Jacobian matrix

$$\frac{\partial(J_{n_0-n^*+2}, H)}{\partial(z_1, \ldots, z_{n_0+1})}$$

must have rank two, otherwise one could write an n_0-point joint invariant I as

$$I(z_1, \ldots, z_{n_0}) = f(J_{n_0-n^*+2}(z_1, \ldots, z_{n_0+1}), H(z_1, \ldots, z_{n_0+1})),$$

a function of $J_{n_0-n^*+2}$ and H, which would contradict the fact that $J_{n_0-n^*+2}$ and H are functionally independent of $\{J_1, \ldots, J_{n_0-n^*+1}\}$, a complete fundamental set of n_0-point joint invariant. The conclusion follows by the implicit function theorem. □

Again, note that we can use the moving frame normalization method to obtain both the invariants J and H. We can use $I_1 = J_{n_0-n^*+2}$ and $I_2 = H$ to parameterize a signature characterizing closed curves or curve segments modulo G. The recognition algorithms are just slightly different that those of Case 1.

Given a curve segment $C = \{\alpha(t)\}$ with n_0 landmarks $(\zeta_1, \ldots, \zeta_{n_0})$ in the direction of the parameterization and the same number of landmarks $(\lambda_1, \ldots, \lambda_{n_0})$

in the other direction, we define its signature as the pair $\Sigma(C) = (\Sigma_+(C), \Sigma_-(C))$, where

$$\Sigma_+(C) = \Big\{ \left(J(\zeta_1, \ldots, \zeta_{n^*}), J(\zeta_2, \ldots, \zeta_{n^*+1}), \ldots, J(\zeta_{n_0-n^*+1}, \ldots, \zeta_{n_0})\right), \\ \left\{J\left(\zeta_{n_0-n^*+2}, \ldots, \zeta_{n_0}, \alpha(t)\right), H\left(\zeta_1, \ldots, \zeta_{n_0}, \alpha(t)\right)\right\}_{0\leq t\leq 1} \Big\}$$

and $\Sigma_-(C) = \Sigma_+(\{\alpha(1-t)\}_{0\leq t\leq 1})$. So each component of the signature is a set containing the string of $n_0 - n^* + 1$ real numbers given by evaluating J on all n^* consecutive landmarks together with the curve parameterized by $J(\zeta_{n_0-n^*+2}, \ldots, \zeta_{n_0}, z)$ and $H(\zeta_1, \ldots, \zeta_{n_0}, z)$ with z varying along the curve. Let $\mathbb{Z}_2$ act on the signature $\Sigma(C)$ by permuting $\Sigma_+(C)$ and $\Sigma_-(C)$.

Theorem 3.14 (For curve segment recognition)**.** *Assuming that $n_0 > n^* + 1$, let J, H and U_{n_0+1} be as defined in Lemma 3.13. Consider two curve segments C and $\bar{C}$ with landmarks $[(\zeta_1, \ldots, \zeta_{n_0}), (\lambda_1, \ldots, \lambda_{n_0})]$ and $[(\bar\zeta_1, \ldots, \bar\zeta_{n_0}), (\bar\lambda_1, \ldots, \bar\lambda_{n_0})]$ respectively. If, for all $0 \leq t \leq 1$, we have*

$$\begin{aligned} \left(\zeta_1, \ldots, \zeta_{n_0}, \alpha(t)\right), \left(\bar\zeta_1, \ldots, \bar\zeta_{n_0}, \bar\alpha(t)\right) &\in U_{n_0+1} \\ \textit{and } \left(\lambda_1, \ldots, \lambda_{n_0}, \alpha(1-t)\right), \left(\bar\lambda_1, \ldots, \bar\lambda_{n_0}, \bar\alpha(1-t)\right) &\in U_{n_0+1}, \end{aligned}$$

then $C \equiv \bar{C} \mod G$ if and only if $\Sigma(C) \equiv \Sigma(\bar{C}) \mod \mathbb{Z}_2$.

Proof. Invariance modulo $\mathbb{Z}_2$ follows from the invariance of the construction up to the choice of direction for the parameterization.

Assume that $\Sigma(C) = \Sigma(\bar{C})$ (reparameterize one of the curve segments if necessary). This implies that the vector value of

$$\begin{gathered} \left(J(\zeta_1, \ldots, \zeta_{n^*}), J(\zeta_2, \ldots, \zeta_{n^*+1}) \ldots, J(\zeta_{n_0-n^*+1}, \ldots, \zeta_{n_0})\right) \\ \text{and } \left(J(\bar\zeta_1, \ldots, \bar\zeta_{n^*}), J(\bar\zeta_2, \ldots, \bar\zeta_{n^*+1}) \ldots, J(\bar\zeta_{n_0-n^*+1}, \ldots, \bar\zeta_{n_0})\right) \end{gathered}$$

are equal. By Lemma 3.12, this means that there exists $g \in G$ such that $g \cdot (\zeta_1, \ldots, \zeta_{n_0}) = (\bar\zeta_1, \ldots, \bar\zeta_{n_0})$. By Lemma 3.13, we have $g \cdot C = \bar{C}$. □

Given a closed curve segments $C = \{\alpha(t)\}$ with $k \geq n_0$ landmarks, we define its signature as $\Sigma(C) = (\Sigma_+(C), \Sigma_-(C))$ where $\Sigma_+(C)$ is the set of k curves

$$\Sigma_+(C) = (\Sigma_{1+}(C), \ldots, \Sigma_{k+}(C))$$

parameterized by

$$\Sigma_{i+} = (J(\zeta_{i+n_0-n^*}, \ldots, \zeta_{n_0+i-1}, z)H(\zeta_i, \ldots, \zeta_{n_0+i-1}, z))$$

for $z \in C$ going from ζ_{n_0+i} to ζ_{n_0+i-1}. As a corollary of 3.14, we have the following theorem.

Theorem 3.15 (For closed curve recognition)**.** *Assuming that $n_0 > n^* + 1$, let J, H and U_{n_0+1} be as defined in Lemma 3.13. Consider two closed (simple) curves $C = \{\alpha(t)\}$ and $\bar{C} = \{\bar\alpha(t)\}$ with $k \geq n_0$ landmarks $(\zeta_1, \ldots, \zeta_k)$ and $(\bar\zeta_1, \ldots, \bar\zeta_k)$ respectively. Set $(\zeta_{k+1}, \ldots, \zeta_{2k}) = (\zeta_1, \ldots, \zeta_k)$ and $(\bar\zeta_{k+1}, \ldots, \bar\zeta_{2k}) = (\bar\zeta_1, \ldots, \bar\zeta_k)$. If, for all $0 \leq t \leq 1$ and all $i = 1, \ldots, k$, we have*

$$\left(\zeta_i, \ldots, \zeta_{n^*+i-1}, \alpha(t)\right), \left(\bar\zeta_i, \ldots, \bar\zeta_{n^*+i-1}, \bar\alpha(t)\right) \in U_{n_0+1},$$

then $C \equiv \bar{C} \mod G$ if and only if $\Sigma_+(C) \equiv \Sigma_+(\bar{C}) \mod \mathbb{Z}_k$ or $\Sigma_+(C) \equiv \Sigma_-(\bar{C}) \mod \mathbb{Z}_k$.

Observe that, in all cases presented, the main tricks used consisted in finding two invariants $I_1(z_1,\ldots,z_n), I_2(z_1,\ldots,z_n)$ such that given $z_1,\ldots,z_{n-1}$, then the last point z_n is uniquely prescribed by the value of $I_1(z_1,\ldots,z_n)$ and $I_2(z_1,\ldots,z_n)$. The natural question to ask, of course, is whether one could build a signature in a similar fashion using invariants that depend on less than n_0+1 points. Unfortunately, there is no way to repeat our trick with such invariants.

Lemma 3.16. *Whenever $n < n_0+1$, there do not exist two invariants $I_1, I_2 : U_n \subset (M^2)^{\times(n)} \to \mathbb{R}$ such that, on U_n, the last point z_n can be expressed as a function*

$$z_n = f\left(z_1,\ldots,z_{n-1}, I_1(z_1,\ldots,z_n), I_2(z_1,\ldots,z_n)\right).$$

Proof. This is because the rank of the Jacobian matrix

$$\mathcal{J} = \frac{\partial(I_1, I_2)}{\partial z_n}$$

is never equal to two on an open and dense subset of any open subset of $(M^2)^{\times(n)}$, since the number $\#_n$ of n-point fundamental invariants is $\#_n - \#_{n-1} = 0$ or 1, but never 2, as discussed in the proof of Lemma 3.8. □

4. Conclusions

Based on a method described in a warm up paper [3] on polygon recognition, we proved the existence of two suitable joint invariants which can be used for parameterizing a signature curve for curve segments or closed curves in a two-dimensional manifold. These two invariants can be obtained by the moving frame method. The signature curve is such that two curves are equivalent modulo a Lie group G if and only if their signature is the same up to the choice of orientation of the curve. We provided a full solution for the group of area preserving planar transformations (equi-affine). Suitable invariants for other important cases, including the Euclidean group and the similarity group (scaling, rotations and translations), are given in the warm up paper. The construction of our signature requires a minimal number of landmarks (n_0, the stabilization order of the prolonged group action on many copies of a manifold), which can themselves be used for testing the equivalence of curves, and thus ruling out the unlikely candidates.

The two main advantages of this method are: the invariants used are more robust than differential invariants and the dimension of the signature is optimal. For lack of space, we did not discuss the use of this signature for G-symmetry detection, although this is just a subproblem of curve recognition. However, a method for symmetry detection in curves can be easily deduced from the method for symmetry detection in polygons described in details in the warm up paper.

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