POLYGON RECOGNITION AND SYMMETRY DETECTION

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ABSTRACT. We introduce an approach based on moving frames for polygon recognition and symmetry detection. We present detailed algorithms for recognition of polygons in $\mathbb{R}^2$ modulo the special Euclidean, Euclidean, equi-affine, skewed-affine and similarity Lie groups. We also solve the case of polygons in the Poincaré half-plane under the action of $SL(2)$ and explain a method applicable to Lie group actions in general. The time complexity of our algorithms is linear in the number of vertices and they are noise resistant. The signatures used allow the detection of partial as well as approximate equivalences.

1. INTRODUCTION

This paper is devoted to equivalence of polygons under Lie group actions. As a subproblem, we also consider symmetries of polygons, which are nothing but self-equivalences, i.e. group transformations leaving the polygon unchanged. We are interested in global, partial and approximate equivalences. The approach we suggest is based on the theory of moving frames ([3], [4], [5]) and constitutes a warm up for a more general method [2] for curve recognition. It consists in constructing a joint invariant signature (JIS) for every polygon. The signature of a polygon in an m-dimensional manifold M is a polygon in $\mathbb{R}^m$ which is the same for all polygons belonging to the same equivalence class; symmetries of a polygon manifest as repetitions in its signature.

Our approach is very simple and fast, but more importantly it is general, having far more applicability than the particular cases we present here. In fact, it can be used for detecting equivalences under any Lie group which acts (locally) effectively on subsets, provided some slight regularity conditions hold. Moreover, it could be generalized to higher dimensional structures such as polyhedra. Recognition and symmetry detection algorithms are already known for a few Lie groups actions (see for example [6, 10, 11]) but, as far as we know, no such general method was published before. To illustrate its power, we show how this method successfully applies to some cases where the Lie group action is less intuitive like the similarity Lie group acting on $\mathbb{R}^2$ and $SL(2)$ acting on the Poincaré half-plane.

One practical advantage of our approach is that it is noise resistant; it can in fact be used for the detection of approximate symmetries. Another advantage is that each point of the signature only depends on a few consecutive points of the polygon. We are in fact able to build signatures which indicate partial equivalences, i.e. when two pieces of a polygon are equivalent. Moreover, the dimension of the signature is optimal and so is the order of complexity of the corresponding detection algorithms. Finally, the modern method of moving frames provides us with effective tools to obtain the invariants we need to parameterize a signature.

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In the following, we shall provide a full solution to the problem of detection of all area preserving affine symmetries in the plane (rotations, reflections, equi-affine and skewed affine transformations). We shall also provide a full solution to the problems of planar polygon recognition modulo the special Euclidean, full Euclidean, equi-affine, skewed-affine and similarity Lie groups, as well as to the problem of polygon recognition in the Poincaré half-plane modulo $SL(2)$. These examples illustrate a method applicable to Lie group actions in general which we shall explain in details.

2. Mathematical Foundations

Let G be a Lie group acting on an m-dimensional smooth (Hausdorff) manifold M.

Definition 2.1. An *invariant* is a real valued function $I : M \to \mathbb{R}$ which remains unchanged under the action of G, more precisely

$$I(g \cdot p) = I(p), \text{ for all } p \in M \text{ and all } g \in G.$$

A *local invariant* is a real valued function $I : M \to \mathbb{R}$ which remains unchanged under the action of a neighborhood N_e of the identity in G,

Definition 2.2. We say that G acts *freely* on M if the identity is the only element of G that fixes any point of M. In other words, if $g \cdot p = p$, for some $g \in G$ and some $p \in M$, then $g = e$, the identity.

Definition 2.3. We say that G acts *semi- regularly* on M if all orbits have the same dimension. If, in addition, any point $p_0 \in M$ is surrounded by an arbitrarily small neighborhood whose intersection with the orbit through p_0 is connected, then we say that G acts *regularly.*

Most of our results are based on the following important theorem. See [7] for a proof.

Theorem 2.4 (Frobenius Theorem)**.** *If G acts on an open set $O \subset M$ semi-regularly with s-dimensional orbits, then $\forall p_0 \in O$ there exist $m - s$ functionally independent local invariants $I_1, \dots, I_{m-s}$ defined on a neighborhood U of p_0 such that any other local invariant H defined near p_0 is a function $H = f(I_1, \dots, I_{m-s})$. If G acts regularly on O, then we can choose $I_1, \dots, I_{m-s}$ to be global invariants on O. In that case, two points $p_1, p_2 \in O$ are in the same orbit relative to G if and only if $I_i(p_1) = I_i(p_2)$, for all $i = 1, \dots, m - s$.*

By *functional independence* of the (smooth) functions $I_1, \dots, I_{m-s}$ on an open set O, we simply mean that the Jacobian matrix of $I_1, \dots, I_{m-s}$ has maximal rank $m - s$ on an open and dense subset of O. The set $\{I_1, \dots, I_{m-s}\}$ is often called a *complete fundamental set of invariants on O.*

The modern theory of moving frames, as developed by Fels and Olver [4, 5], defines a (left) moving frame as follows.

Definition 2.5. A *moving frame* is a map $\rho : M \to G$ such that $\rho(g \cdot p) = g \cdot \rho(p)$, $\forall p \in M$, $\forall g \in G$. A *local moving frame* is a map $\rho : M \to G$ such that $\rho(g \cdot p) = g \cdot \rho(p)$, $\forall p \in M$, $\forall g \in N_e \subset G$, for some neighborhood N_e of the identity $e \in G$.

The conditions of existence of a moving frame are very precise.

Theorem 2.6. *A (local) moving frame exists in a neighborhood of a point $p_0 \in M$ if and only if G acts (locally) freely and regularly near p_0.*

Definition 2.7. We say that G acts on M *effectively* if

$$\{g \in G| \quad g \cdot p = p, \text{ for all } p \in M\} = \{e\}.$$

We say that G acts on M *locally effectively* if

$$\{g \in G| \quad g \cdot p = p, \text{ for all } p \in M\}$$

is a discrete subset of G.

Definition 2.8. We say that G acts *effectively on subsets of* M if, for any open subset $U \subset M$,

$$\{g \in G| \quad g \cdot p = p, \text{ for all } p \in U\} = \{e\}.$$

We say that G acts *locally effectively on subsets of* M if, for any open subset $U \subset M$,

$$\{g \in G| \quad g \cdot p = p, \text{ for all } p \in M\}$$

is a discrete subset of G.

For analytic Lie group actions, effectiveness implies effectiveness on subsets. However this is not true for merely smooth Lie group actions.

A moving frame can be used as a tool to compute a complete fundamental set of invariants. The moving frame normalization method, described in details in [5], is an algorithmic way to do so. Throughout this paper, we shall assume that the reader is familiar with that method.

Let $M^{\times(n)} := M \times M \times \ldots \times M$ (n times) be the Cartesian product of n copies of the manifold M. In the case where the action is not (locally) free, one option is to prolong the action of G on $M^{\times(n)}$ by setting $g \cdot (p_1, \ldots, p_k) = (g \cdot p_1, \ldots, g \cdot p_k)$, for all $g \in G$ and $(p_1, \ldots, p_k) \in M^{\times(k)}$, and hope that the action then becomes free.

Let δ be the dimension of G. The following important result was proved in [1].

Theorem 2.9. *If G acts (locally) effectively on subsets of M, then there exists a minimal integer n_0 such that, for all integers $n \geq n_0$, G acts locally freely on an open and dense subset of $M^{\times(n)}$.*

Definition 2.10. Let $n \in \mathbb{N}$. An *(n-point) joint invariant* is an invariant of the prolonged action of G on $M^{\times(n)}$.

Let H be a finite group acting on M whose action commutes with that of G. The following theorem will be needed shortly.

Proposition 2.11. *Let M be an m-dimensional manifold and $z = (z_1, \ldots, z_m)$ be local coordinates near $z_0 \in N$. Suppose that, in a neighborhood of z_0, the action of G is regular with s-dimensional orbits and the action of H is continuous and free. Then there exist a neighborhood U of z_0 and functionally independent $J_1, \ldots, J_{m-s} : U \to \mathbb{R}$ which are invariant under both G and H and such that two points $z_1 \in h_1 \cdot U$ and $z_2 \in h_2 \cdot U$, for some $h_1, h_2 \in H$, are in the same orbit relative to $G \times H$ if and only if $J_i(z_1) = J_i(z_2)$, for all $i = 1, \ldots, m - s$.*

Proof. By freeness and continuity of the action of H, we can choose U so that H maps U to a finite number of disjoint open sets U_i, with $i = 1, \ldots, |H|$. By hypothesis, we can also choose U so that the action of G is regular with s-dimensional orbits on U. Since the actions of G and H commute, the group H maps orbits of G to orbits of G. Moreover, the dimension of the orbits is preserved under the group action of H (because composition of any group transformation with its inverse must give back the identity). So H maps s-dimensional orbits of G in U to s-dimensional orbits of G in the U_i's. In addition, continuity of the action of H

insures that connectedness is preserved by H, so the action of G must be regular on each U_i.

Let $\pi : M \to M/H$ be the projection map $\pi(x) = x \mod H$. Consider the induced action of G on M/H. More precisely, consider the action of G on $U \mod H$, which has a manifold structure as a subset of M/H. The map $\pi : U_i \to U \mod H$ is a diffeomorphism, for all $i = 1, \dots, |H|$. Since our choice of U guarantees that the action of G on $U \mod H$ is regular and s-dimensional, then by Theorem 2.4 there exist $I_1, \dots, I_{m-s} : U \mod H \to \mathbb{R}$ functionally independent invariants (under G).

Define $J_i : M \to \mathbb{R}$ by $J_i = I_i \circ \pi$ for $i = 1, \dots, m - s$. By construction, all J_i's are invariant under both G and H. Moreover, functional independence is preserved by the mere change of coordinates π. This shows the first part of the statement. The second part follows from Theorem 2.4. □

3. A signature for global polygon recognition

3.1. **Equivalence of ordered sets of points under a Lie group action.** In what follows, we will keep writing M for an m-dimensional manifold and G for a δ-dimensional Lie group acting on M. Let $(p_1, \dots, p_k)$ be a point of $M^{\times(k)}$. Suppose that G acts on a neighborhood $U_k \subset M^{\times(k)}$ of $(p_1, \dots, p_k)$ regularly with s-dimensional orbits. Then by Theorem 2.4, there exist fundamental invariants $I_1, \dots, I_{km-s} : U_M \subset U_k \to \mathbb{R}$. The map $S_M : U_M \subset M^{\times(k)} \to \mathbb{R}^{km-s}$ defined by

$$S_M(p_1, \dots, p_k) = \begin{pmatrix} I_1(p_1, \dots, p_k) \\ \vdots \\ I_{km-s}(p_1, \dots, p_k) \end{pmatrix}$$

is a signature for ordered sets of k points, i.e. it maps all $(p_1, \dots, p_k) \in U_M$ to a representative of the orbit through $(p_1, \dots, p_k)$.

Theorem 3.1. *Let $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ be two points of $U_M \subset M^{\times(k)}$. There exists $g \in G$ such that $g \cdot (p_1, \dots, p_k) = (q_1, \dots, q_k)$ if and only if*

$$S_M(p_1, \dots, p_k) = S_M(q_1, \dots, q_k).$$

Proof. By Theorem 2.4. □

So from the value of a finite set of invariants, one can conclude whether two ordered sets of points are equivalent. This provides us with a fast and easy way to recognize ordered sets of points up to a Lie group action.

3.2. **Equivalence of polygons under a Lie group action.** On a complete manifold, polygons are usually defined as a sequence of points (the vertices) linked by geodesics (the edges). Two polygons are equivalent if they can be mapped onto each other by a group transformation $g \in G$. This implies that g must map the geodesics of one polygon to the geodesics of the other. Observe that, on a complete manifold for which any two points can be linked by a unique length minimizing geodesic, two polygons are equivalent under the action of a geodesic preserving group if and only if their vertices are equivalent under the group. Our method for polygon recognition consists in testing solely for the equivalence of the vertices. In that context, we shall define polygons simply in terms of their vertices. Doing so, we shall have a generalization of the concept of polygon to manifolds that are not necessarily complete and thus consider a more general problem.

Let the cyclic group $\mathbb{Z}_k$ act on $M^{\times(k)}$ by permuting the k points cyclically. Let $\pi \in \mathbb{Z}_2$ act on $M^{\times(k)}$ by reversing the order of the k points, i.e. $\pi(p_1, p_2, \ldots, p_k) = (p_k, \ldots, p_2, p_1)$, for all $(p_1, p_2, \ldots, p_k) \in M^{\times(k)}$. Together, $\mathbb{Z}_k$ and $\mathbb{Z}_2$ generate a group $\mathbb{H}_k =< \mathbb{Z}_k, \mathbb{Z}_2 >$ acting on $M^{\times(k)}$. Observe that $\mathbb{Z}_k$ and $\mathbb{Z}_2$ do not commute. However, the action of $\mathbb{H}_k$ is quite simple.

Lemma 3.2. *If $h \in \mathbb{H}_k$, then either*

$$h \in \mathbb{Z}_k, \tag{1}$$

or

$$h = c \cdot \pi, \text{ with } c \in \mathbb{Z}_k. \tag{2}$$

Let $(p_1, \ldots, p_k)$ be a point of $M^{\times(k)}$. Let $\mathfrak{P}^k = M^{\times(k)}/\mathbb{Z}_k$ be the set of k ordered points in $M^{\times(k)}$ modulo the action of $\mathbb{Z}_k$. If $(p_1, \ldots, p_k) \in M^{\times(k)}$, the corresponding point in $\mathfrak{P}^k$ will be written as $[p_1, \ldots, p_k]$. The action of G on $M^{\times(k)}$ naturally induces an action of G on $\mathfrak{P}^k$, namely $g \cdot [p_1, \ldots, p_k] = [g \cdot p_1, \ldots, g \cdot p_k]$, for $g \in G$ and $p_1, \ldots, p_k \in M$.

Let $\mathcal{P}^k = M^{\times(k)}/\mathbb{H}_k$ be the set of k ordered points in M modulo the action of $\mathbb{H}_k$. If $(p_1, \ldots, p_k) \in M^{\times(k)}$, the corresponding point in $\mathcal{P}^k$ will be written as $\langle p_1, \ldots, p_k \rangle$. The action of G on $M^{\times(k)}$ naturally induces an action of G on $\mathcal{P}^k$, namely $g \cdot \langle p_1, \ldots, p_k \rangle = \langle g \cdot p_1, \ldots, g \cdot p_k \rangle$, for all $g \in G$ and $p_1, \ldots, p_k \in M$.

Definition 3.3. A *k-vertex polygon*, or *k-gon*, is a point $\langle p_1, \ldots, p_k \rangle$ of $\mathcal{P}^k$.

Definition 3.4. We say that two k-gons $P = \langle p_1, \ldots, p_k \rangle$ and $Q = \langle q_1, \ldots, q_k \rangle$ are *equivalent under* G if there exist $g \in G$ such that $g \cdot \langle p_1, \ldots, p_k \rangle = \langle q_1, \ldots, q_k \rangle$. In that case, we write $P \equiv Q \mod G$.

Definition 3.5. We say that a polygon $P = \langle p_1, \ldots, p_k \rangle$ *has a non-trivial G-symmetry* if there exists $g \in G \setminus \{e\}$ such that $g \cdot \langle p_1, \ldots, p_k \rangle = \langle p_1, \ldots, p_k \rangle$.

In other words, if a polygon $P = \langle p_1, \ldots, p_k \rangle$ has a non-trivial G-symmetry, then there exists $g \in G \setminus \{e\}$ and $h \in \mathbb{H}_k$ such that $g \cdot (p_1, \ldots, p_k) = h \cdot (p_1, \ldots, p_k)$. According to Lemma 3.2, either $h \in \mathbb{Z}_k$ or $h = c \cdot \pi$, with $c \in \mathbb{Z}_k$ and $\pi \in \mathbb{Z}_2$ as defined above. Similarly if $P = \langle p_1, \ldots, p_k \rangle$ is equivalent to $Q = \langle q_1, \ldots, q_k \rangle$ modulo G, then $g \cdot (p_1, \ldots, p_k) = h \cdot (q_1, \ldots, q_k)$ for some $g \in G$ and $h \in \mathbb{Z}_k$ or $h = c\pi$ with $c \in \mathbb{Z}_k$. These simple facts will be used in our polygon recognition algorithms later on.

If G acts regularly with s-dimensional orbits on some open set $U \subset M^{\times(k)}$, then by Theorem 2.4 , in a certain neighborhood U_M of any point $(p_1, \ldots, p_k) \in U$, there exists a complete fundamental set of invariants $\{I_1, \ldots, I_{mk-s}\} : U_M \to \mathbb{R}$ under G. Assuming that H_k acts freely on U (which can be guaranteed by taking $p_1, \ldots, p_k$ distinct for example), then by Proposition 2.11, there also exists $\{\bar{I}_1, \ldots, \bar{I}_{mk-s}\}$, a complete fundamental set of invariants under $G \times \mathbb{Z}_k$, as well as $\{\tilde{I}_1, \ldots, \tilde{I}_{mk-s}\}$, a complete fundamental set of invariants under $G \times \mathbb{H}_k$, all defined on some neighborhood of $(p_1, \ldots, p_k)$. Define $S_{\mathfrak{M}} : U_{\mathfrak{M}} \subset \mathfrak{M}^k \to \mathbb{R}^{km-s}$ and $S_{\mathcal{P}} : U_{\mathcal{P}} \subset \mathcal{P}^k \to \mathbb{R}^{km-s}$ by

$$S_{\mathfrak{M}}([p_1, \ldots, p_k]) = \begin{pmatrix} \bar{I}_1(p_1, \ldots, p_k) \\ \vdots \\ \bar{I}_{mk-s}(p_1, \ldots, p_k) \end{pmatrix},$$

$$S_{\mathcal{P}}(\{p_1,\ldots,p_k\}) \quad = \quad \begin{pmatrix} \tilde{I}_1(p_1,\ldots,p_k) \\ \vdots \\ \tilde{I}_{mk-s}(p_1,\ldots,p_k) \end{pmatrix}.$$

The maps $S_{\mathfrak{M}}$, $S_{\mathcal{P}}$ and S_M can be used as signatures.

Theorem 3.6. *Let $P = \langle p_1,\ldots,p_k\rangle$ and $Q = \langle q_1,\ldots,q_k\rangle$ be two k-gons. Assume that $(p_1,\ldots,p_k) \in U_M$, $[p_1,\ldots,p_k] \in U_{\mathfrak{M}}$ and that $\langle p_1,\ldots,p_k\rangle \in U_{\mathcal{P}}$. Then $P \equiv Q$* mod G

$$\begin{aligned} \Leftrightarrow S_M(p_1,\ldots,p_k) \quad &= \quad S_M(h\cdot(q_1,\ldots,q_k)) \textit{ for some } h \in \mathbb{H}_k \\ \Leftrightarrow S_{\mathfrak{M}}([p_1,\ldots,p_k]) \quad &= \quad S_{\mathfrak{M}}([q_1,\ldots,q_k]) \\ &\textit{or} \\ S_{\mathfrak{M}}([p_1,p_2,\ldots,p_k]) \quad &= \quad S_{\mathfrak{M}}([q_k,\ldots,q_2,q_1]) \\ \Leftrightarrow S_{\mathcal{P}}(\langle p_1,\ldots,p_k\rangle) \quad &= \quad S_{\mathcal{P}}(\langle q_1,\ldots,q_k\rangle) \end{aligned}$$

Proof. The necessity of the first and third statements follow from the invariance of the signature. For the second statement, we also use Lemma 3.2.

To prove sufficiency of the first statement, assume that $S_1(p_1,\ldots,p_k) = S_1(h\cdot(q_1,\ldots,q_k))$. Then by Proposition 2.11, there exists $g \in G$ such that $g\cdot(p_1,\ldots,p_k) = h\cdot(q_1,\ldots,q_k)$. So $g\cdot\{p_1,\ldots,p_k\} = \{q_1,\ldots,q_k\}$.

To prove sufficiency of the second statement, assume that $S_2([p_1,\ldots,p_k]) = S_2([z\cdot(q_1,\ldots,q_k)])$ for some $z \in \mathbb{Z}_2$. By Proposition 2.11, this means that there exist $g \in G$ and $c \in \mathbb{Z}_k$ such that

$$g\cdot(p_1,\ldots,p_k) = c\cdot(z\cdot q_1,\ldots,z\cdot q_k),$$

and therefore $g\cdot\{p_1,\ldots,p_k\} = \{q_1,\ldots,q_k\}$. The proof for sufficiency of the third statement is similar. □

3.3. **Equivalence of point configurations under Lie group action.** Let $\mathbb{S}_k$ be the symmetric group. The elements of $\mathbb{S}_k$ act on $\{(p_1,\ldots,p_k) \in M^{\times(k)}\}$ by permuting the points $p_1,\ldots,p_k$.

Definition 3.7. A k-point configuration is a point of $M^{\times(k)}/\mathbb{S}_k$.

In other words, a k-point configuration on M is a finite set of k points on M which are not ordered in any way. We shall use the notation $|p_1,\ldots,p_k|$ for the k-point configurations corresponding to $p_1,\ldots,p_k \in M$. The action of G on M naturally induces an action of G on k-point configurations. The previous polygon recognition method is easy to extend to point configuration recognition. In fact, we can repeat the same arguments as before to claim the existence of a complete set of fundamental invariants $\hat{I}_1,\ldots,\hat{I}_{mk-s} : \hat{U} \to \mathbb{R}$ under $G \times \mathbb{S}_k$. Also, the map $\hat{S} : \hat{U} \subset M^{\times(k)}/\mathbb{S}_k \to \mathbb{R}^{km-s}$ defined by

$$\hat{S}(|p_1,\ldots,p_k|) = \begin{pmatrix} \hat{I}_1(p_1,\ldots,p_k) \\ \vdots \\ \hat{I}_{mk-s}(p_1,\ldots,p_k) \end{pmatrix}$$

is a signature for k-point configurations.

Theorem 3.8. *Let $|p_1, \ldots, p_k|$ and $|q_1, \ldots, q_k| \in \hat{U}$ be two k-point configurations. Then there exists $g \in G$ such that $g \cdot |p_1, \ldots, p_k| = |q_1, \ldots, q_k|$*

$$\Leftrightarrow \hat{S}(|p_1, \ldots, p_k|) = \hat{S}(|q_1, \ldots, q_k|).$$

We can go even further of course. For example, we can consider a finite number of polygons (without any order). In a similar manner, we can define a signature which will characterize these polygons up to a Lie group action. This can be used for recognizing objects made of a finite number of disconnected polygons.

4. A signature for partial polygon recognition

The previous section presented a way to recognize polygons globally. However, we are also interested in the case where a piece of a polygon is equivalent to a piece of another polygon. In particular, we would like the signature to indicate whether two pieces of polygons are equivalent under a group action, or if a polygon has a certain symmetry. This would be a complex task using the previous signatures. In the following, we explain a simpler method.

Recall that m is the dimension of the manifold M. We would like to parameterize a signature with no more than m invariants, since this would be the optimal number. We will explain shortly how to choose suitable invariants. But first let us give some definitions.

Let $n, k \in \mathbb{N}$ with $k \geq n$. Given m invariants $I_1, \ldots, I_m : M^{\times(n)} \to \mathbb{R}$ of the action of G on $M^{\times(n)}$ and $(p_1, \ldots, p_k) \in M^{\times(k)}$, define

$$I_{i,r} : M^{\times(k)} \to \mathbb{R}, \text{ for } i = 1, \ldots m \text{ and } r = 1, \ldots, k,$$

$$\text{by } I_{i,r}(p_1, \ldots, p_k) = I_i(p_r, p_{r+1}, \ldots, p_{r+n-1}).$$

setting $p_{k+1} = p_1, \quad p_{k+2} = p_2, \quad \ldots, \quad p_{k+n-1} = p_{n-1}$. For example, if $I_1(p_1, p_2, p_3) = |p_3 - p_2|$, then

$$\begin{array}{rcccl} I_{1,1}(p_1, \ldots, p_k) & = & I_1(p_1, p_2, p_3) & = & |p_3 - p_2|, \\ I_{1,2}(p_1, \ldots, p_k) & = & I_1(p_2, p_3, p_4) & = & |p_4 - p_3|, \\ I_{1,3}(p_1, \ldots, p_k) & = & I_1(p_3, p_4, p_5) & = & |p_5 - p_4|, \\ \vdots & & \vdots & & \vdots \\ I_{1,k-1}(p_1, \ldots, p_k) & = & I_1(p_{k-1}, p_k, p_1) & = & |p_1 - p_k|, \\ I_{1,k}(p_1, \ldots, p_k) & = & I_1(p_k, p_1, p_2) & = & |p_2 - p_1|. \end{array}$$

Define $S : M^{\times(k)} \to (\mathbb{R}^m)^{\times(k)}$ by

$$S(p_1, \ldots, p_k) = \begin{pmatrix} I_{1,1}(p_1, \ldots, p_k) & , \ldots, & I_{1,k}(p_1, \ldots, p_k) \\ \vdots & & \vdots \\ I_{m,1}(p_1, \ldots, p_k) & , \ldots, & I_{m,k}(p_1, \ldots, p_k) \end{pmatrix}. \tag{3}$$

Let $\mathbb{Z}_k$ act on $S(p_1, \ldots, p_k)$ by permuting its columns. If we let $\mathfrak{S}$ be the map $\mathfrak{S} : \mathfrak{P}^k \to (\mathbb{R}^m)^{\times(k)} \bmod \mathbb{Z}_k$ given by

$$\mathfrak{S}([p_1, \ldots, p_k]) = S(p_1, \ldots, p_k) \bmod \mathbb{Z}_k,$$

then the following diagram commutes.

$$\begin{array}{ccc} M^{\times(k)} & \xrightarrow{S} & (\mathbb{R}^m)^{\times(k)} \\ \text{mod } \mathbb{Z}_k \downarrow & & \downarrow \text{mod } \mathbb{Z}_k \\ \mathfrak{P}^k & \xrightarrow[\mathfrak{S}]{} & (\mathbb{R}^m)^{\times(k)} \quad \text{mod } \mathbb{Z}_k \end{array}$$

The maps S and $\mathfrak{S}$ can be used as signatures in the following instance.

Theorem 4.1 (For global recognition). *Let $P = \langle p_1, \ldots, p_k \rangle$ and $Q = \langle q_1, \ldots, q_k \rangle$ be two k-gons. Assume that the set $\{I_{1,r}, \ldots, I_{m,r}\}_{r=1}^{k}$ contains a complete fundamental set of k-point joint invariants defined on an open set $U_k \subset M^{\times(k)}$ on which G acts regularly and that $(p_1, \ldots, p_k), (q_1, \ldots, q_k) \in U_k$. Then $P \equiv Q$ modulo G*

$$\begin{aligned} \Leftrightarrow \quad S(p_1, \ldots, p_k) &= S(h \cdot (q_1, \ldots, q_k)), \text{ for some } h \in \mathbb{H}_k, \\ \Leftrightarrow \quad \mathfrak{S}([p_1, \ldots, p_k]) &= \mathfrak{S}([q_1, \ldots, q_k]) \\ &\quad or \\ \mathfrak{S}([p_1, \ldots, p_k]) &= \mathfrak{S}([q_k, \ldots, q_1)]) \end{aligned}$$

Proof. This follows from Theorem 3.6. □

Corollary 4.2 (For symmetry detection.). *Let $P = \langle p_1, \ldots, p_k \rangle$ be a polygon. Assume that the invariants $\{I_{1,r}, \ldots, I_{m,r}\}_{r=1}^{k}$ contain a complete fundamental set of k-point joint invariants defined on an open set $U_k \subset M^{\times(k)}$ on which G acts regularly and that $\mathbb{H}_k \cdot (p_1, \ldots, p_k) \subset U_k$. Then P has a non-trivial G-symmetry if and only if*

$$S(p_1, \ldots, p_k) = S(h \cdot (p_1, \ldots, p_k)), \text{ for some } h \in \mathbb{H}_k \setminus \{e\}.$$

We will explain shortly how to construct m suitable invariants $I_1, \ldots, I_m$, suitable in the sense that $\{I_{1,r}, \ldots, I_{m,r}\}_{r=1}^{k}$ contains a complete fundamental set of k-point joint invariants on some open set. But before we present the general method for polygon recognition, let us consider two instructive examples.

5. An example of an orientation preserving Lie group action

5.1. **Construction of the signature.** As a first example, consider $SE(2)$, the group of orientation preserving rigid motions in the plane (rotations and translations). We call it the *special Euclidean group*. It is generally accepted to call the corresponding symmetries of polygons *rotational symmetries*, since any such symmetry corresponds to a rotation around some interior point of the polygon.

This well known result can be proved [9] using the moving frame method.

Theorem 5.1. *For $SE(2)$ acting on $\mathbb{R}^2$, we have the following.*

(1) *There are no one-point joint invariants.*
(2) *There is one fundamental two-point joint invariants $J^0(p_1, p_2) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, namely $J^0(p_1, p_2) = |p_2 - p_1|$.*
(3) *There are three fundamental three-point joint invariants*

$$J_1^1(p_1, p_2, p_3), \quad J_2^1(p_1, p_2, p_3), \quad J_3^1(p_1, p_2, p_3) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},$$

namely

$$\begin{aligned} J_1^1(p_1,p_2,p_3) &= |p_2 - p_1|, \\ J_2^1(p_1,p_2,p_3) &= |p_3 - p_2|, \\ J_3^1(p_1,p_2,p_3) &= \Delta_{123} := \frac{1}{2}\det[p_3 - p_1, p_2 - p_1]. \end{aligned}$$

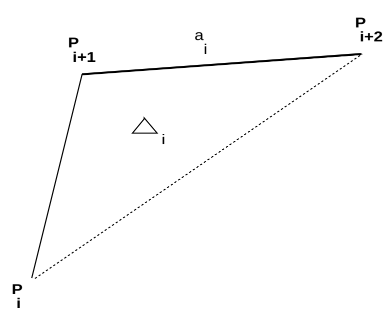
We are looking for two suitable joint invariants I_1 and I_2 to build a signature. Again, by suitable we mean that $\{I_{1,r}, I_{2,r}\}_{r=1}^k$ should contain a complete set of fundamental k-point joint invariants on some open set. For that, we can take $I_1(p_1,p_2,p_3) = |p_3 - p_2|$ and $I_2(p_1,p_2,p_3) = \Delta_{123}$, the signed area of the triangle with vertices p_1, p_2 and p_3. This is a natural choice, according to our general method to be explained later (see Theorem 7.17).

Given a k-gon $P = \langle p_1,\dots,p_k\rangle$, we choose an orientation and a starting point for P, given by say $\{p_1,\dots,p_k\}$, and we define its special Euclidean joint invariant signature ($SEJIS$) as the sequence of k points given by

$$SEJIS(p_1,\dots,p_k) = \{a_i, \Delta_i\}_{i=1}^k, \tag{4}$$

where $a_i = |p_{i+2} - p_{i+1}|$ and $\Delta_i := \Delta_{i(i+1)(i+2)}$ is the signed area given by the determinant of the 2×2 matrix $\frac{1}{2}[p_{i+2} - p_i, p_{i+1} - p_i]$. See the illustration in Figure 1. The reasons why the invariants a and Δ can be used to build a signature are

FIGURE 1



contained in two properties.

First, when evaluating a_i and Δ_i for $i = 1,\dots,k$, one obtains (among other things) all fundamental joint invariants which only depend on the first two points p_1 and p_2. In this particular case, there is only one, namely $J^0(p_1,p_2) = |p_2 - p_1|$. In other words, we have

$$\{|p_2 - p_1|\} \subset \{a_i, \Delta_i\}_{i=1}^k$$

with $\{|p_2 - p_1|\}$ a complete fundamental set of two-point (p_1,p_2) joint invariants. This guarantees the first property called *two-point projectability* ($\star$) which we shall define precisely a bit further. The idea behind demanding this property is to be able to use the fact that, if $J^0(p_1,p_2) = J^0(q_1,q_2)$, then there exists $g \in SE(2)$ such that $g\cdot q_1 = q_1$ and $g\cdot p_2 = q_2$, so that equality of the signatures will imply that one can map the first two points of the respective polygons together by a Euclidean transformation.

Secondly, given p_1 and p_2 with $p_1 \neq p_2$, then p_3 is uniquely determined by the value of $a_1 = |p_3 - p_2|$ and $\Delta_1 = \frac{1}{2}\det[p_3 - p_1, p_2 - p_1]$. In other words, p_3 is a function

$$p_3 = f(p_1, p_2, a_1, \Delta_1),$$

provided that $p_1 \neq p_2$. In fact, p_{i+2} is a function $p_{i+2} = f(p_i, p_{i+1}, a_i, \Delta_i)$ for all i's, whenever $p_{i+1} \neq p_i$. This guarantees the (soon to be defined) second property called *third point reductivity* ($\star\star$) when consecutive points are distinct. Demanding this property insures that any transformation g which maps the first two points (p_1, p_2) to (q_1, q_2) must also map the rest of the polygons together, provided that their signature is the same.

As will be proved in Theorem 7.10, $n = 3$ is the minimal number of points for which we can find two n-point joint invariants I_1 and I_2 which are $(n-1)$-point projectable and n^{th} point reductive on some open set. Any two other invariants $\tilde{I}_1$, $\tilde{I}_2$ satisfying ($\star$) and ($\star\star$) are guaranteed to encode all the relevant information contained in a polygon modulo $SE(2)$ and thus to parameterize a signature $\tilde{S} = \{\tilde{I}_{1,r}, \tilde{I}_{1,r}\}_{r=1}^{k}$ containing a complete fundamental set of k-point joint invariants.

Definition 5.2. Choose an orientation (i.e. a traveling direction on the vertices) for $P = \langle p_1, \ldots, p_k \rangle$. The *special Euclidean joint invariant signature curve* ($SEJIS$ curve) of P with respect to this orientation is the piecewise linear curve obtained by joining the points of the signature which correspond to consecutive vertices of the polygon by a straight oriented line segment.

The $SEJIS$ curve represents the special Euclidean signature up to cyclic permutations of its k points. This takes care of the ambiguity about the starting point p_1. There remains one ambiguity: the traveling direction. In fact, the points of the $SEJIS$ are *not* invariant under the action of reversing the order of the vertices of the polygon. However, if we restrict ourselves to simple polygons, i.e. polygons whose edges do not cross each other, then one can choose an orientation (clockwise for example) and this orientation shall not be affected by the action of $SE(2)$. In fact the $SEJIS$ curve characterizes all simple polygons.

Theorem 5.3 (For simple polygon recognition modulo $SE(2)$)**.** *Two planar polygons whose edges do not cross and whose vertices are labeled clockwise (counterclockwise) are equivalent under the action of $SE(2)$ if and only if their $SEJIS$ curve with respect to the clockwise (counterclockwise) orientation is the same.*

Proof. Since the points of the signatures are functions of the basic $SE(2)$-invariants, they are $SE(2)$-invariant themselves. Moreover the order of the vertices is chosen in an invariant way, except for the starting point. Therefore if two polygons are equivalent under the action of $SE(2)$, then their signature will be identical, up to cyclic permutation. Now suppose that $P = \langle p_1, \ldots, p_k \rangle$ and $Q = \langle q_1, \ldots, q_k \rangle$ are two polygons with the same $SEJIS = (s_1, \ldots, s_k)$. Assume that s_1 corresponds to (p_1, p_2, p_3) and (q_1, q_2, q_3), that s_2 corresponds to (p_2, p_3, p_4) and (q_2, q_3, q_4), and so on. Since the signature of the two polygons is the same, we have $|p_2 - p_1| = |q_2 - q_1|$ (by ($\star$)). So we can find $g \in SE(2)$ which maps p_1 to q_1 and p_2 to q_2. Moreover since p_{i+2} is uniquely prescribed by p_i, p_{i+1} and the value of a_i and Δ_i (by ($\star\star$)), we have that g also maps p_3 to q_3, and p_4 to q_4, and so on. Therefore $g \cdot P = Q$. $\square$

However, in practice we do not need to restrict ourselves to simple polygons. All we have to do in order to characterize non-simple polygons is to use our very same $SEJIS$ curve while taking into account the fact that we might have chosen a different orientation and starting point.

Theorem 5.4 (For polygon recognition modulo $SE(2)$). *Two planar polygons $P = \langle p_1, \ldots, p_k \rangle$ and $Q = \{q_1, \ldots, q_k\}$ with distinct consecutive vertices are equivalent under the action of $SE(2)$*

$$\Leftrightarrow SEJIS(p_1, \ldots, p_k) = SEJIS(h \cdot (q_1, \ldots, q_k)), \text{ for some } h \in \mathbb{H}_k.$$

Unfortunately, the fact that we only characterize polygons up to $\mathbb{H}_k$ is inherent to the construction of the signature. However, Lemma 3.2 facilitates the search for a possible $h \in \mathbb{H}_k$. In facts, since the $SEJIS$ commutes with rotations, we have the following useful lemma.

Lemma 5.5. *Let $SEJIS(p_1, \ldots, p_k) = (s_1, \ldots, s_k)$, $SEJIS(q_1, \ldots, q_k) = (\sigma_1, \ldots, \sigma_k)$, and $SEJIS(q_k, \ldots, q_1) = (\bar{\sigma}_k, \ldots, \bar{\sigma}_1)$. Then $SEJIS(p_1, \ldots, p_k) = SEJIS(h \cdot (q_1, \ldots, q_k))$ for some $h \in \mathbb{H}_k$ if and only if*

$$\begin{array}{rcl} (s_1, \ldots, s_k) & = & c \cdot (\sigma_1, \ldots, \sigma_k) \\ & \text{or} & \\ (s_1, \ldots, s_k) & = & c \cdot (\bar{\sigma}_k, \ldots, \bar{\sigma}_1) \end{array}$$

for some $c \in \mathbb{Z}_k$.

Since a symmetry is a self-equivalence, we can modify the previous theorem in order to detect symmetries. In fact in this case, the ambiguity about the direction is waived and the orientation of the polygon can be chosen arbitrarily.

Theorem 5.6 (For $SE(2)$-symmetry detection). *If $P = \langle p_1, \ldots, p_k \rangle$ is a planar polygon and $SEJIS(p_1, \ldots, p_k) = (s_1, \ldots, s_k)$, then P has an f-fold rotational symmetry if and only if*

$$(s_{\frac{k}{f}+1}, \ldots, s_k, s_1, \ldots, s_{\frac{k}{f}}) = (s_1, \ldots, s_k),$$

in other words, if and only if the signature curve winds f times on itself.

Proof. The polygon P has an f-fold symmetry if and only if there exists $g \in SE(2)$ such that $g \cdot (p_1, \ldots, p_k) = (p_{\frac{k}{f}+1}, \ldots, p_k, p_1, \ldots, p_{\frac{k}{f}})$. By invariance of the signature, this means that

$$\begin{array}{rcl} SEJIS(p_1, \ldots, p_k) & = & SEJIS(p_{\frac{k}{f}+1}, \ldots, p_k, p_1, \ldots, p_{\frac{k}{f}}) \\ \Leftrightarrow (s_1, \ldots, s_k) & = & (s_{\frac{k}{f}+1}, \ldots, s_k, s_1, \ldots, s_{\frac{k}{f}}), \end{array}$$

which proves the necessity of the statement.

Now if $(s_1, \ldots, s_k) = (s_{\frac{k}{f}+1}, \ldots, s_k, s_1, \ldots, s_{\frac{k}{f}})$, then by property ($\star$) and ($\star\star$), there exists $g \in SE(2)$ such that $g \cdot (p_1, \ldots, p_k) = (p_{\frac{k}{f}+1}, \ldots, p_k, p_{k+1}, \ldots, p_{\frac{k}{f}})$. This proves sufficiency. □

So a k-gon P has an f-fold symmetry if and only if the signature curve is traced f times in the same direction as one travels along the curve. This can be checked in $O(k)$ by a computer. We implemented the algorithm using the software MATLAB and computed the results for a number of examples. One of them is the four-branch star of Figure 5. For a counterclockwise orientation of this octagon, the program gives the following $SEJIS$ (rounded to 4 digits).

$$SEJIS = \begin{bmatrix} 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 \\ 4 & -3 & 4 & -3 & 4 & -3 & 4 & -3 \end{bmatrix}$$

The $SEJIS$ curve, represented in Figure 5, is obtained by joining these points with a straight oriented segment. Although the polygon has eight vertices, the graph of the signature shows only two vertices: the signature curve winds four times on itself. This reflects the fact that the four-branch star shown has a four-fold symmetry.

5.2. **Advantages of this** $SEJIS$**.** This is clearly not the only way to build a signature. So why do we prefer this method to others?

First of all, this signature indicates whether two pieces of polygons are the same up to rigid motions (partial equivalences as defined below). This is because the invariants used depend on very few points and they are chosen so that partial equivalences correspond to specific similarities of the signature curves. More precisely, we have the following theorem.

Theorem 5.7 (For partial equivalences modulo $SE(2)$)**.** *Let $P = \langle p_1, \ldots, p_k \rangle$ be a planar polygon with distinct consecutive vertices and $SEJIS(p_1, \ldots, p_k) = (s_1, \ldots, s_k)$. Let $Q = \langle q_1, \ldots, q_l \rangle$ be another planar polygon with distinct consecutive vertices and $SEJIS(q_1, \ldots, q_k) = (\sigma_1, \ldots, \sigma_l)$. Let $n \in \mathbb{N}$, let $\tilde{P} = (p_i, \ldots, p_{i+n})$ and let $\tilde{Q} = (q_j, \ldots, q_{j+n})$. Let s_i^1 be the first component of s_i, namely $|p_{i+2} - p_{i+1}|$, and similarly for σ_i^1.*

- *For n=1. There exists $g \in SE(2)$ such that $g \cdot (p_i, p_{i+1}) = (q_j, q_{j+1})$ if and only if $s_{i-1}^1 = s_{j-1}^1$.*
- *For $n > 1$. There exist $g \in SE(2)$ such that $g \cdot (p_i, \ldots, p_{i+n}) = (q_j, \ldots, q_{j+n})$ if and only if $(s_i, \ldots, s_{i+n-2}) = (\sigma_j, \ldots, \sigma_{j+n-2})$ and $s_{i-1}^1 = s_{j-1}^1$.*

We call the $(n+1)$ consecutive vertices of P given by $\tilde{P} = (p_i, \ldots, p_{i+n})$ a *piece of* P. If a piece of P is equivalent to a piece of Q, more precisely if

$$\begin{array}{rcll} & (p_i, \ldots, p_{i+n}) & \equiv & (q_k, q_{k+1}, \ldots, q_{j+n}) \mod G \\ \text{or} & (p_i, \ldots, p_{i+n}) & \equiv & (q_{j+n}, \ldots, q_{k+1}, q_k) \mod G, \end{array}$$

then we say that P is *partially equivalent* to Q.

It is easy to modify our method in order to recognize what we call *polygonal segments* (or *open polygons*). Given an ordered set of points $(p_1, \ldots, p_k)$ in the plane, define its signature as the ordered set of points given by

$$SEJIS_{open}(p_1, \ldots, p_k) = \{(|p_2 - p_1|, 0)\} \cup \{a_i, \Delta_i\}_{i=1}^{k-2}.$$

Corollary 5.8 (For open polygon recognition)**.** *There exists $g \in SE(2)$ such that $g \cdot (p_1, \ldots, p_k) = (q_1, \ldots, q_k)$ if and only if*

$$SEJIS_{open}(p_1, \ldots, p_k) = SEJIS_{open}(q_1, \ldots, q_k).$$

Another advantage of our signature is that it is noise resistant. We are using joint invariants and, in general, the value of such invariants does not change much when the points are slightly perturbed (as opposed to say differential invariants). In fact in this specific case, if we measure each point p_i as $\tilde{p}_i$, and if the noise is such that the measures are within a certain error say

$$\tilde{p}_i = p_i \pm (\epsilon, \epsilon),$$

then one can verify that the measured signature $\{\tilde{a}_i, \tilde{\Delta}_i\}_{i=1}^k$ will have an error bounded by:

$$\begin{aligned} \tilde{a}_i &= a_i \pm 2\sqrt{2}\epsilon \\ \tilde{\Delta}_i &= \Delta_i \pm \frac{\sqrt{2}}{2}(|p_{i+1} - p_i| + |p_{i+2} - p_i| + |p_{i+2} - p_{i+1}|)\epsilon. \end{aligned}$$

6. An example of a non-orientation preserving Lie group action

A slightly more complicated case is the recognition of planar polygons up to rotations and reflections. The corresponding group is called the (full) Euclidean group $E(2)$ and consists in all rigid motions in the plane: translations, rotations and reflections.

The following theorem can be obtained using the moving frame normalization method.

Theorem 6.1. *For $E(2)$ acting on $\mathbb{R}^2$, the situation is as follows.*

(1) *There are no one-point joint invariants.*
(2) *There is one fundamental two-point joint invariants $J^0(p_1, p_2) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, namely $J^0(p_1, p_2) = |p_2 - p_1|$.*
(3) *There are three fundamental three-point joint invariants*

$$J_1^1(p_1, p_2, p_3), \quad J_2^1(p_1, p_2, p_3), \quad J_3^1(p_1, p_2, p_3) : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},$$

namely

$$\begin{aligned} J_1^1(p_1, p_2, p_3) &= |p_2 - p_1|, \\ J_2^1(p_1, p_2, p_3) &= |p_3 - p_2|, \\ J_3^1(p_1, p_2, p_3) &= \frac{1}{2}|\det[z_3 - z_1, z_2 - z_1]|, \text{ the area of the triangle} \\ &=: |\Delta_{123}|. \end{aligned}$$

Again we are interested in finding two joint invariants I_1 and I_2 such that $\{I_{1,r}, I_{2,r}\}_{r=1}^k$ contains a complete fundamental set of k-point invariants. In other words, we want I_1 and I_2 to *record* all the information contained in the polygon modulo $E(2)$. We proceed similarly as for $SE(2)$ to construct a Euclidean joint invariant signature ($EJIS$ for short). According to our general method (see Theorem 7.17), the invariants that are naturally prescribed by the results of our normalization are

$$I_1(p_1, p_2, p_3) = |p_3 - p_2| \qquad \text{and} \qquad I_2(p_1, p_2, p_3) = |\Delta_{123}|.$$

These two invariants are such that

$$\{|p_2 - p_1|\} \subset \{I_1(p_i, p_{i+1}, p_{i+2}), I_2(p_i, p_{i+1}, p_{i+2})\}_{i=1}^k$$

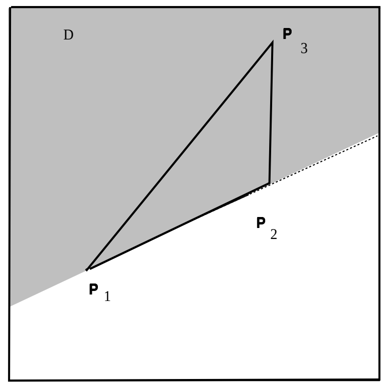
with $\{|p_2 - p_1|\}$ a complete set of fundamental invariants only depending on the first two points p_1 and p_2. This guarantees the first property called two-point projectability ($\star$).

We also have that given $p_1, p_2, p_3 \in D = \{\Delta_{123} \geq 0\}$, then p_3 is uniquely determined by the value of $I_1(p_1, p_2, p_3)$ and $I_2(p_1, p_2, p_3)$. In other words, we have

$$p_3 = f(p_1, p_2, I_1(p_1, p_2, p_3), I_2(p_1, p_2, p_3))$$

for $p_1, p_2, p_3 \in D$. This guarantees the second property, called third point reductivity ($\star\star$) on the restricted domain D. See Figure 2 for an illustration.

FIGURE 2



So we can use I_1 and I_2 to parameterize a signature for convex polygons for example, but not for all polygons. This is due to the domain restriction on $(\star\star)$.

Since it is desirable to characterize all polygons, we would like to find a way around that difficulty. What we need is to find invariants for which $(\star\star)$ holds on a bigger domain. But since any three-point invariant is a function of J_1^1, J_2^1 and J_3^1 and, for any p_1 and p_2, there are two choices of p_3 which lead to the same value of J_1^1, J_2^1 and J_3^1, there is no hope to build a signature on a bigger domain using only three-point joint invariants. We need to use at least four-point joint invariants.

Theorem 6.2. [9] *All three-point invariants of $E(2)$ acting on the plane are functions of the distances $|p_i - p_j|$, for $i, j = 1, 2, 3$ and $i < j$.*

All four-point invariants of $E(2)$ acting on the plane are functions of the distances $|p_i - p_j|$, for $i, j = 1, 2, 3, 4$ and $i < j$.

Observe that the fundamental three-point joint invariants written here are different than those of Theorem 6.1. This illustrates the non-uniqueness of fundamental sets of invariants.

According to Theorem 6.2, in order to have three-point projectability, it is enough that the signature contain the invariants

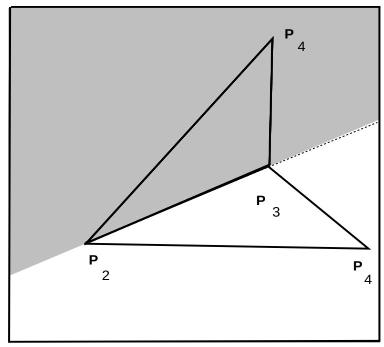
$$|p_2 - p_1|, \quad |p_3 - p_2| \quad \text{and} \quad |p_3 - p_1|.$$

This way, if the signature of $(p_1, \ldots, p_k)$ is the same as the signature of $(q_1, \ldots, q_k)$, then we can map (p_1, p_2, p_3) to (q_1, q_2, q_3) with a Euclidean transformation. For example $J_1 = |p_4 - p_3|$ and $J_2 = |p_4 - p_2|$ would do.

In order to have fourth point reductivity, we need to choose two four-point joint invariants $I_1(p_1, p_2, p_3, p_4)$ and $I_2(p_1, p_2, p_3, p_4)$ whose values uniquely prescribe p_4, given p_1, p_2 and p_3. If we take $I_1 = |p_4 - p_3|$ and $I_2 = |p_4 - p_2|$ then (of course!) there still remains some ambiguity about the position of p_4 (as illustrated in Figure 3).

What we need is to know the sign of the triangle defined by p_2, p_3 and p_4. So we look for a four-point joint invariant which, given p_1, p_2 and p_3, determines the sign of this triangle. Observe that $\text{sign}(\Delta_{234})$ itself is *not* an invariant. However, $\text{sign}(\Delta_{123}\Delta_{234})$ is an invariant and, provided $\Delta_{123} \neq 0$ and $\Delta_{234} \neq 0$, satisfies our

FIGURE 3



requirement. In fact, the invariants

$$I_1(p_1,p_2,p_3,p_4) = |p_4 - p_3| \quad \text{and} \quad I_2(p_1,p_2,p_3,p_4) = \text{sign}(\Delta_{123}\Delta_{234})|p_4 - p_2|$$

can be used to parameterize a Euclidean joint invariant signature

$$EJIS(p_1,\ldots,p_k) = \{I_{1,r}, I_{2,r}\}_{r=1}^{k}$$

for polygons for which no three consecutive points lie on a straight line. This is because the two fundamental three-point joint invariants $|p_2 - p_1|$, $|p_3 - p_2|$, and $|p_3 - p_1|$ are functions

$$\begin{aligned} |p_2 - p_1| &= f_1(\{I_{1,r}, I_{2,r}\}_{r=k-2}^{1}), \\ |p_3 - p_2| &= f_2(\{I_{1,r}, I_{2,r}\}_{r=k-2}^{1}), \\ |p_3 - p_1| &= f_3(\{I_{1,r}, I_{2,r}\}_{r=k-2}^{1}). \end{aligned}$$

This guarantees property $(\star)$ called *three-point projectability*

Moreover, given p_i, p_{i+1} and p_{i+2}, then p_{i+3} is uniquely determined by $I_{1,i}$ and $I_{1,i}$. This guarantees property $(\star\star)$ called fourth point reductivity.

Theorem 6.3 (For polygon recognition modulo $E(2)$)**.** *Two planar polygons $P = \langle p_1,\ldots,p_k\rangle$ and $Q = \langle q_1,\ldots,q_k\rangle$ which contain no three consecutive vertices lying on a straight line are equivalent under the action of $E(2)$*

$$\begin{aligned} \Leftrightarrow EJIS(p_1,\ldots,p_k) &= EJIS(h\cdot(q_1,\ldots,q_k)), \text{ for some } h \in \mathbb{H}_k, \\ \Leftrightarrow EJIS(p_1,\ldots,p_k) &= c\cdot EJIS(q_1,\ldots,q_k) \\ & \quad or \\ EJIS(p_1,\ldots,p_k) &= c\cdot EJIS(q_k,\ldots,q_1) \text{ for some } c \in \mathbb{Z}_k \end{aligned}$$

Proof. By invariance of the functions chosen to parameterize it, the $EJIS$ of two equivalent polygons must be the same, modulo the choice of starting point and direction. This proves the necessity of the first statement. To prove necessity of the second statement, we use Lemma 3.2 and the fact that $EJIS$ commutes with rotations.

If $EJIS(p_1,\ldots,p_k) = EJIS(h\cdot(q_1,\ldots,q_k))$, let $(\tilde{q}_1,\ldots,\tilde{q}_k) = h\cdot(q_1,\ldots,q_k)$. Property $(\star)$ allows us to conclude that $\exists g \in G$ such that $g\cdot(p_1,p_2,p_3) = (\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)$. Property $(\star\star)$ implies that $g\cdot(p_1,\ldots,p_k) = (\tilde{q}_1,\ldots,\tilde{q}_k)$. and therefore $P \equiv Q$

mod $E(2)$. This proves the sufficiency of the first statement. The proof of the sufficiency of the second statement is similar. □

Euclidean symmetries are of two types: rotations, which are the orientation preserving symmetries, and reflections, which are the orientation reversing symmetries. Both types of symmetries are indicated by the signature, although in general they cannot be distinguished. However, for simple polygons (i.e. when the edges do not cross each other) it is possible to distinguish both types of symmetries.

Theorem 6.4 (For orientation preserving $E(2)$-symmetry detection in simple polygons). *If $P = (p_1, \dots, p_k)$ is any simple planar polygon which contains no three consecutive vertices lying on a straight line and $EJIS(p_1, \dots, p_k) = (s_1, \dots, s_k)$, then P has a f-fold rotational symmetry, $1 < f \leq k$, if and only if*

$$(s_{\frac{k}{f}+1}, \dots, s_k, s_1, \dots, s_{\frac{k}{f}}) = (s_1, \dots, s_k),$$

that is to say, if and only if the signature curve winds f times on itself.

Proof. For simple polygons, rotations are the only $E(2)$ transformations which preserve the traveling direction on the vertices, since they are the only transformations which preserve orientation. So the proof is the same as for $SE(2)$ symmetries. □

Theorem 6.5 (For orientation reversing $E(2)$-symmetry detection in simple polygons). *Let $P = \langle p_1, \dots, p_k \rangle$ be any simple planar polygon which contains no three consecutive vertices lying on a straight line. Then P has an axis of reflection if and only if $EJIS(p_1, \dots, p_k) = c \cdot EJIS(p_k, \dots, p_1)$, for some $c \in \mathbb{Z}_k$.*

More precisely, P has an axis of reflection passing through the vertex p_1 if and only if

$$EJIS(p_1, \dots, p_{k-1}, p_k) = EJIS(p_1, p_k, p_{k-1}, \dots, p_2),$$

and P has an axis of reflection passing in the middle of the edge joining the vertex p_1 to p_2 if and only if

$$EJIS(p_1, p_2, \dots, p_{k-1}, p_k) = EJIS(p_2, p_1, p_k, \dots, p_3).$$

Proof. For simple polygons, rotations are the only $E(2)$-symmetries which reverse the traveling direction on the vertices, since they are the only transformations which reverse orientation. By invariance of the $EJIS$ and since the $EJIS$ commutes with rotations, if $g \cdot (p_1, \dots, p_k) = c \cdot (p_k, \dots, p_1)$ for some $c \in \mathbb{Z}_k$, then $EJIS(p_1, \dots, p_k) = c \cdot EJIS(p_k, \dots, p_1)$.

Now if $EJIS(p_1, \dots, p_k) = c \cdot EJIS(p_k, \dots, p_1)$, then ($\star$) and ($\star\star$) imply that there exists $g \in G$ such that $g \cdot (p_1, \dots, p_k) = c \cdot (p_k, \dots, p_1)$. In particular, if $c \cdot (p_k, \dots, p_1) = (p_1, p_k, \dots, p_2)$, then p_1 is fixed so we have an axis of reflection passing through p_1. Also if $c \cdot (p_k, \dots, p_1) = (p_2, p_1, p_k, \dots, p_3)$, then p_1 is mapped to p_2 and p_2 is mapped to p_1, so we have an axis of reflection passing through the middle of the edge joining p_1 to p_2. □

We implemented this algorithm using the software MATLAB and computed the results for a few examples. One of them is the four branch star of Figure 5. The program gives the following $EJIS$ (rounded to 4 digits) for one direction.

$$EJIS1 = \begin{bmatrix} 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 \\ -4.2426 & -2 & -4.2426 & -2 & -4.2426 & -2 & -4.2426 & -23 \end{bmatrix}$$

For the other direction, we obtained the following.

$$EJIS2 = \begin{bmatrix} 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 & 2.236 \\ -2 & -4.2426 & -2 & -4.2426 & -2 & -4.2426 & -23 & -4.2426 \end{bmatrix}$$

The $EJIS$ curves, represented in Figure 5, are obtained by joining those points with a straight oriented segment. Again the winding number is four, i.e. this polygon has a four fold rotational symmetry. We also detected four axes of symmetries which are also graphed on the figure.

In general we have the following.

Theorem 6.6 (For E(2)-symmetry detection)**.** *If $P = \langle p_1, \ldots, p_k \rangle$ is any planar polygon (not necessarily simple) for which no three consecutive vertices lie on a straight line, then P has a non-trivial $E(2)$ symmetry if and only if there exists $h \in H_k \setminus \{e\}$ such that*

$$EJIS(p_1, \ldots, p_k) = EJIS(h \cdot (p_1, \ldots, p_k)).$$

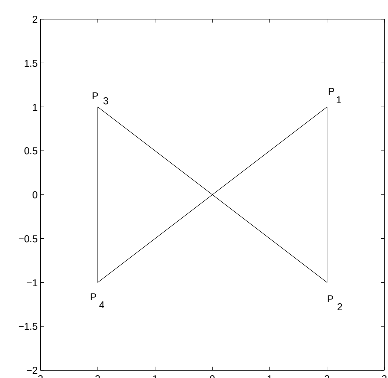
Consider the following instructive example.

Example 6.7. Let

$$P = \langle p_1, \ldots, p_4 \rangle = \langle (2, 1), (2, -1), (-2, 1), (-2, -1) \rangle.$$

Observe that this planar polygon is not simple since two edges cross at the origin. (See Figure 4.) It has an axis of reflection which passes through the y-axis, and the

FIGURE 4



corresponding symmetry maps

$$(p_1, p_2, p_3, p_4) \quad \text{to} \quad (p_3, p_4, p_1, p_2).$$

Although this is an orientation reversing symmetry, the traveling direction on the vertices is preserved.

Also, this polygon has a two-fold rotational symmetry which maps

$$(p_1, p_2, p_3, p_4) \quad \text{to} \quad (p_4, p_3, p_1, p_2).$$

So there is an orientation preserving symmetry which reverses the traveling direction on the vertices.

This example, brought to our attention by Peter Olver, illustrates the fact that in general, the $EJIS$ does not distinguish orientation preserving and reversing symmetries. However, using the results of the previous section, it is easy to determine which symmetries are rotations, and which symmetries are not. All one has to do is to use the result provided by both the $EJIS$ and the $EJIS$ to identify which $E(2)$-symmetries are not $SE(2)$-symmetries: these are the reflections.

7. Construction of a G-invariant signature curve

Now that we have an intuitive idea of how to build a signature curve, we are ready to describe the general method. This method will help us construct JIS for less intuitive Lie groups. As illustrations, the examples of the equi-affine, skewed affine and similarity groups acting on the plane, as well as the action of $SL(2)$ on the Poincaré half-plane will be presented in the last sections.

7.1. Two Sufficient Properties.

Our goal is to characterize k-gons in a m-dimensional manifold M up to the action of an δ-dimensional Lie group G. By Theorem 4.1, all we need to do is to find m n-point joint invariants $I_1, \ldots, I_m$ such that $\{I_{1,r}, \ldots, I_{m,r}\}_{r=1}^{k}$ contains a complete fundamental set of k-point joint invariants on some open subset of $M^{\times(k)}$. We shall do this by demanding that the joint invariants satisfy two properties, which we call $(n-1)$-point projectability and n^{th} point reductivity on ab open set. For short, we will often denote these properties by $(\star)$ and $(\star\star)$ respectively. It will soon become clear to the reader that $(\star)$ and $(\star\star)$ are merely one of many ways to achieve our goal of obtaining suitable invariants. But although these conditions are stronger than needed, they happen to be satisfied by the output of a simple constructive algorithm.

In order to simplify the following discussion, we introduce some notations. Let $n \in \mathbb{N}$, let $1 \leq i_1 < i_2 < \ldots < i_R \leq n$ and let $\Pi_{(n;i_1,i_2,\ldots,i_R)} : M^{\times(n)} \to M^{\times(R)}$ be the restriction on the $i_1, \ldots, i_R$ factors, i.e.

$$\Pi_{(n;i_1,i_2,\ldots,i_R)}(p_1, \ldots, p_n) = (p_{i_1}, p_{i_2}, \ldots, p_{i_R}).$$

For any subset $U_n \subset M^{\times(n)}$, denote by $\Pi_{(n;i_1,i_2,\ldots,i_R)}U_n$ the image of U_n under $\Pi_{(n;i_1,i_2,\ldots,i_R)}$. We denote by $\Pi^*_{(n;i_1,i_2,\ldots,i_R)}f$ the pull-back under $\Pi_{(n;i_1,i_2,\ldots,i_R)}$ of any function $f : O \subset M^{\times(R)} \to \mathbb{R}$. For example, if $f : \mathbb{R}^{\times(2)} \to \mathbb{R}$ is given by $f(x_1, x_2) = x_1 + x_2$, then $\Pi^*_{(3;2,3)}f(x_1, x_2, x_3) = x_2 + x_3$.

Now, let $l, n \in \mathbb{N}$ with $l \geq n$ and let $U_n \subset N^{\times(n)}$. Denote by $U_n|^l$ the subset of $M^{\times(l)}$ defined by

$$U_n|^l = \bigcap_{i=0}^{l-n} M^{\times(i)} \times U_n \times M^{\times(l-n-i)},$$

and by $U_n|^{<l>}$ the one defined by

$$U_n|^{<l>} = \bigcap_{c_i \in \mathbb{Z}_l} c_i \cdot \left(U_n \times M^{\times(l-n)}\right).$$

For example, if $U_3 \subset (\mathbb{R}^2)^{\times(3)}$ is the set of triples of points (p_1, p_2, p_3) such that the area $\Delta(p_1, p_2, p_3) \neq 0$, then $U_3|^4$ is the set of quadruples of points (p_1, p_2, p_3, p_4) such that $\Delta(p_1, p_2, p_3) \neq 0$ and $\Delta(p_2, p_3, p_4) \neq 0$, whereas $U_3|^{<4>}$ is the set of quadruples of points (p_1, p_2, p_3, p_4) such that $\Delta(p_1, p_2, p_3) \neq 0$, $\Delta(p_2, p_3, p_4) \neq 0$, $\Delta(p_3, p_4, p_1) \neq 0$ and $\Delta(p_4, p_1, p_2) \neq 0$.

Given an n-point joint invariant $I_n : U_n \to \mathbb{R}$, we can construct two $(n+1)$-point joint invariants J_1, J_2 on $U_n|^{n+1}$ by setting $J_1 = \Pi^*_{(n+1;1,\ldots,n)} I_n$ and $J_2 = \Pi^*_{(n+1;2,\ldots,n+1)} I_n$. This notation will come handy in the following as such constructions will be used repeatedly in order to obtain suitable invariants. The notation $U_n|^{<l>}$ will mostly be used with $l = k$, the number of vertices of a polygon, in order to designate k-gons for which all n consecutive vertices lie in an open set U_n (as in Lemma 7.3 for example). This will allow us to successively evaluate a set of n-point joint invariants $I_1, \ldots, I_m : U_n \to \mathbb{R}$ on all consecutive n points of the k-gon in order to obtain the signature as defined by Equation 3.

Observe that $U_n|^{<l>} \subset U_n|^l$ and that both sets can potentially be empty. However, by symmetry of the Cartesian action of G on $M^{\times(n)}$, we will be able to simply replace U_n by $\tilde{U}_n = \bigcup_{c_i \in \mathbb{Z}_n} c_i \cdot O_n$ for some open subset $O_n \subset U_n$. By taking the union of all cyclic permutations, we are guaranteed that $\tilde{U}_n\Big|^l \supset \tilde{U}_n\Big|^{<l>} \neq \emptyset$.

Let $\bar{m} \in \mathbb{N}$ (perhaps but not necessarily equal to m) and let $c_{ir} \in \mathbb{R}$, for $i = 1, \ldots, \bar{m}$ and $r = 1, \ldots, k$. Denote by C the matrix $C = \{c_{ir}\}$. Given $\bar{m}$ invariants $I_1, \ldots, I_{\bar{m}} : U_n \subset M^{\times(n)} \to \mathbb{R}$, we define the set $L^k_C = L^k_C(\{I_{1,r}, \ldots, I_{\bar{m},r}\}^k_{r=1})$ by

$$L^k_C = \{(p_1, \ldots, p_k) \in U_n|^{<k>} \quad \text{such that} \quad I_{i,r}(p_1, \ldots, p_k) = c_{ir}, \quad \text{for} \quad i = 1, \ldots, \bar{m} \quad \text{and} \quad r = 1, \ldots, k\} \subset M^{\times(k)}.$$

The set L^k_C is called a *level set* of $\{I_{1,r}, \ldots, I_{\bar{m},r}\}^k_{r=1}$.

The first property that we will demand is that, for any $k \geq n$, all level sets $L^k_C \subset M^{\times(k)}$ project down to subsets of $M^{\times(n-1)}$ which locally correspond to orbits of the action of G on $M^{(n-1)}$. More precisely, we demand the following.

Definition 7.1. Let $\bar{m}, n \in \mathbb{N}$ with $n > 1$. We say that $\bar{m}$ n-point joint invariants $I_1, \ldots, I_{\bar{m}} : U_n \subset M^{\times(n)} \to \mathbb{R}$ are $(n-1)$*-point projectable* $(\star)$ on U_n if for any $k \geq n$ and for any $C \in \mathbb{R}^{\bar{m} \times n}$ with $L^k_C = L^k_C(\{I_{1,r}, \ldots, I_{\bar{m},r}\}^k_{r=1}) \neq \emptyset$, the set $\Pi_{(k;1,\ldots,n-1)} L^k_C$ can be written as

$$\Pi_{(k;1,\ldots,n-1)} L^k_C = U \cap O$$

with U an open subset of $M^{\times(n-1)}$ and O an orbit of G acting on $M^{\times(n-1)}$.

A different way to look at $(n-1)$-point projectability is the following.

Lemma 7.2. *Let $U_n \subset M^{\times(n)}$ and assume that G acts regularly on $\Pi_{(n;1,2,\ldots,n-1)} U_n$. The n-point joint invariants $I_1, \ldots, I_{\bar{m}} : U_n \to \mathbb{R}$ are $(n-1)$-point projectable on U_n if and only if for any $k \geq n$, the pull back $\Pi^*_{(k;1,\ldots,n-1)} J$ of any invariant J defined on $\Pi_{(1,\ldots,n-1)} U_n$ is a function of the invariants $\left\{I_{1,r}, \ldots, I_{\bar{m},r} : U_n|^{<k>} \to \mathbb{R}\right\}^k_{r=1}$.*

Proof. This follows from Theorem 2.4. □

The reason why we demand $(n-1)$-point projectability is shown in the next lemma.

Lemma 7.3. *Let the invariants $I_1, \ldots, I_{\bar{m}} : U_n \to \mathbb{R}$ be $(n-1)$-point projectable on $U_n \subset M^{\times(n)}$ and assume that G acts regularly on $\Pi_{(n;1,2,\ldots,n-1)} U_n$. For $k \geq n$, consider the map $S : U_n|^{<k>} \to \mathbb{R}^{\bar{m} \times k}$ defined by Equation 3. Let $P = \langle p_1, \ldots, p_k \rangle$ and $Q = \langle q_1, \ldots, q_k \rangle$ be two polygons such that $(p_1, \ldots, p_k), (q_1, \ldots, q_k) \in U_n|^{<k>}$. If $S(p_1, \ldots, p_k) = S(q_1, \ldots, q_k)$, then there exists $g \in G$ such that $g \cdot (p_1, \ldots, p_{n-1}) = (q_1, \ldots, q_{n-1})$.*

Proof. This follows from Theorem 2.4. □

The second property we demand is the following, which can only hold for a number of invariants $\bar{m} = m$.

Definition 7.4. The n-point joint invariants $I_1, \dots, I_m : U_n \subset M^{\times(n)} \to \mathbb{R}$ are said to be *n^{th} point reductive on U_n* if given any $(p_1, \dots, p_{n-1}) \in \Pi_{(n;1,\dots,n-1)} U_n$ and real numbers $c_i \in I_i(U_n)$ for $i = 1, \dots, m$, then there exists a unique $p_n \in \Pi_{(n;n)} U_n$ such that $I_i(p_1, \dots, p_n) = c_i$, for all $i = 1, \dots, m$.

In other words, we ask that $p_n \in \Pi_{(n;n)} U_n$ be a function

$$p_n = f(p_1, \dots, p_{n-1}, I_1(p_1, \dots, p_n), \quad \dots \quad , I_m(p_1, \dots, p_n)).$$

Together, properties $(\star)$ and $(\star\star)$ will guarantee that $I_1, \dots, I_m$ record all the relevant information about a polygon, i.e. all the information contained in the polygon modulo G. We thus define the following.

Definition 7.5. We say that m n-point joint invariants are *recorders* on $U_n \subset M^{\times(n)}$ if they are both $(n-1)$-point projectable $(\star)$ and n^{th} point reductive $(\star\star)$ on U_n.

Proposition 7.6. *Let $U_n \subset M^{\times(n)}$ be an open set and assume that G acts on $\Pi_{(n;1,\dots,n-1)} U_n$ regularly. If $I_1, \dots, I_m$ are recorders on U_n, then for any $k \geq n$, the set $\{I_{1,r}, \dots, I_{m,r}\}_{r=1}^{k}$ contains a complete fundamental set of invariants on $U_n|^{<k>}$.*

Proof. Let s_{n-1} denote the orbit dimension of the action of G on $\Pi_{(n;1,\dots,n-1)} U_n$. By $(\star)$ and Theorem 2.4, we have

$$\text{rank } \frac{\partial \{I_{1,r}, \dots, I_{m,r}\}_{r=1}^{n-1}}{\partial(p_1, \dots, p_{n-1})} = m(n-1) - s_{n-1}$$

on an open and dense subset of $U_n|^{<k>}$. Observe that, by $(\star\star)$, we also have

$$\text{rank } \frac{\partial \{I_{1,n+i}, \dots, I_{m,n+i}\}}{\partial(p_{n+i})} = m, \text{ for all } i = 0, \dots, k-n,$$

on an open and dense subset of $U_n|^{<k>}$. Therefore, on an open and dense subset of $U_n|^{<k>}$, we have

$$\text{rank } \frac{\partial \{I_{1,r}, \dots, I_{m,r}\}_{r=1}^{k}}{\partial(p_1, \dots, p_k)} \geq m(n-1) - s_{n-1} + m(k-n+1) = mk - s_{n-1}.$$

However, the number of k-point fundamental invariants is less than or equal to $mk - s_{n-1}$. So there must be exactly $(mk - s_{n-1})$ k-point fundamental invariants[1] and we must have equality in the previous inequation. □

Corollary 7.7 (For global recognition modulo G.)**.** *Let $U_n \subset M^{\times(n)}$ be an open set such that G acts on $\Pi_{(n;1,\dots,n-1)} U_n$ regularly and let $I_1, \dots, I_m$ be recorders on U_n. Let $k \geq n$ and consider $S : U_n|^{<k>} \to (\mathbb{R}^m)^{\times(k)}$ as defined by Equation 3. Two polygons $P = \langle p_1, \dots, p_k \rangle$ and $Q = \langle q_1, \dots, q_k \rangle$ such that $(p_1, \dots, p_k), (q_1, \dots, q_k) \in U_n|^{<k>}$ are equivalent modulo G if and only if*

$$S(p_1, \dots, p_k) = S(q_1, \dots, q_k) \mod \mathbb{H}_k.$$

[1]See Corollary 7.18.

The converse of 7.6 is not true as illustrated by the following examples. Take G to be the special Euclidean group acting on the plane. Let p_1, p_2, and p_3 be three consecutive points on a polygon. Then the signed area of the triangle defined by p_1, p_2, and p_3 together with the distance between p_2, and p_3 satisfy ($\star$) and ($\star\star$) and therefore can be used to recognize polygons modulo orientation preserving rigid motions. However, the signature given by the signed area of the triangle defined by p_1, p_2, and p_3 together with the distance between p_1, and p_2 does not satisfy ($\star\star$) but obviously still generates a complete fundamental set of k-point invariants for any $k \geq 3$.

Although n^{th} point reductivity is not necessary, it is an easy condition to satisfy, as will be shown later. Moreover the inverse function theorem provides a simple test for making sure this property is locally satisfied. Finally, it is a very natural property to require for detecting partial equivalences in polygons.

Theorem 7.8 (For partial recognition modulo G)**.** *Let $U_n \subset M^{\times(n)}$ be an open set such that G acts on $\Pi_{(n;1,\ldots,n-1)}U_n$ regularly. Let $I_1,\ldots,I_m$ be n-point joint invariants which are n^{th} point projectable on U_n and let $J_1,\ldots,J_N$ be a complete fundamental set of invariants on $\Pi_{(n;1,\ldots,n-1)}U_n$. Let $p_1,\ldots,p_l$ be $l \geq n$ consecutive vertices of a polygon P and $q_1,\ldots,q_l$ be l consecutive vertices of a polygon Q. There exists $g \in G$ such that $g \cdot (p_1,\ldots,p_l) = (q_1,\ldots,q_l)$ if and only if*

$$J_j(p_1,\ldots,p_{n-1}) = J_j(q_1,\ldots,q_{n-1}), \text{ for all } j = 1,\ldots,N \tag{5}$$

and

$$\begin{aligned} I_i(p_1,\ldots,p_n) &= I_i(q_1,\ldots,q_n), \\ I_i(p_2,\ldots,p_{n+1}) &= I_i(q_2,\ldots,q_{n+1}) \\ &\vdots \\ I_i(p_{l-n},\ldots,p_l) &= I_i(q_{l-n},\ldots,q_l), \text{ for all } i = 1,\ldots,m \end{aligned} \tag{6}$$

Proof. By invariance of the I's and J's, "$\Rightarrow$" is true.
To show "$\Leftarrow$", assume $J_i(p_1,\ldots,p_{n-1}) = J_i(q_1,\ldots,q_{n-1})$, for $i = 1,\ldots,N$. Then by Theorem 2.4, there exists $g \in G$ such that $g\cdot(p_1,\ldots,p_{n-1}) = (q_1,\ldots,q_{n-1})$. By ($\star\star$), condition (6) implies that $g \cdot (p_1,\ldots,p_l) = (q_1,\ldots,q_l)$. ☐

If $I_1,\ldots,I_m$ are m joint invariants which are recorders on $U_n \subset M^{\times(n)}$, then the signature S as defined by Equation 3 can be used for partial recognition or partial symmetry detection whenever G acts regularly on $\Pi_{(n;1,\ldots,n-1)}U_n$. Indeed, ($\star$) guarantees that the pull back by $\Pi_{(k;1,\ldots,n-1)}$ of a complete fundamental set of invariants $J_1,\ldots,J_N : \Pi_{(n;1,\ldots,n-1)}U_n \to \mathbb{R}$ yields k-point joint invariants $\Pi^*_{(k;1,\ldots,n-1)}J_1,\ldots,\Pi^*_{(k;1,\ldots,n-1)}J_N$ which can be expressed as functions

$$\Pi^*_{(k;1,\ldots,n-1)}J_i = f_i(\{I_{1,r},\ldots,I_{m,r}\}_{r=1}^k), \text{ for } i = 1,\ldots,N.$$

Therefore the value of $J_1,\ldots,J_n$ can be determined from the signature. One can thus determine whether condition (5) is satisfied by looking at the signatures. Condition (6) is indicated by a partial superposition of the signatures. So both conditions can be checked given the signatures.

7.2. **Construction of recorders $I_1,\ldots,I_m$.** In this section, we will determine how and in what circumstances moving frames can be used to construct m n-point joint invariants which are recorders on some open set.

Assume that the δ-dimensional Lie group G acts (locally) effectively on subsets of M. Denote by s_n the maximal orbit dimension of the prolonged action of G on $M^{\times(n)}$. Let n_o be the minimal integer such that for all $n \geq n_0$, the maximal orbit dimension $s_n = \delta$, the dimension of G. We call n_0 the stabilization order. By Theorem 2.9, such an integer exists.

Lemma 7.9. *Let $n \in \mathbb{N}$. Assume that G acts regularly on both an open set $U_{n+1} \subset M^{\times(n+1)}$ and its restriction $\Pi_{(n+1;1,\ldots,n)}U_{n+1}$. Let $J_1, \ldots, J_N$ be a complete fundamental set of invariants on $\Pi_{(n+1;1,\ldots,n)}U_{n+1}$. Then, in a neighborhood $\tilde{U}_{n+1}$ of any point $z^{(n+1)} \in U_{n+1}$, there exist R invariants $I_1, \ldots, I_R$ such that*

$$\{\Pi^*_{(n+1;1,\ldots,n)}J_1, \ldots, \Pi^*_{(n+1;1,\ldots,n)}J_N, I_1, \ldots, I_R\}$$

is a complete fundamental set of invariants on $\tilde{U}_{n+1}$. If $n \geq n_0$, then $R = m$. Otherwise $R < m$. In any case, these R invariants can be obtained by the (partial) moving frame normalization method.

Proof. By Theorem 2.4, there are exactly $nm - s_n$ fundamental n-point joint invariants and exactly $(n+1)m - s_{n+1}$ fundamental $(n+1)$-point joint invariants. The difference is

$$(n+1)m - s_{n+1} - (nm - s_n) = m + s_n - s_{n+1}.$$

Let $R = m + s_n - s_{n+1}$. Observe that $R < m$, unless $s_n = s_{n+1}$. It is shown in [1] that $s_n = s_{n+1}$ if and only if $n \geq n_0$. So if $n < n_0$, then R is strictly smaller than m, otherwise $R = m$.

For the construction of $I_1, \ldots, I_R$, let $z^{(n+1)} \in U_{n+1}$, and let us first assume that $n+1 \geq n_0$. Then we can build a local moving frame $\rho(p_1, \ldots, p_{n+1})$ in a neighborhood of $z^{(n+1)}$. Consider the group action equation $\bar{p}_{n+1} = g \cdot p_{n+1}$. As explained in details in [8], setting $g = \rho(p_1, \ldots, p_{n+1})$ into this equation gives

$$\bar{p}_{n+1}|_{g=\rho(p_1,\ldots,p_n)} = (I_1, \ldots, I_m),$$

a vector made of m $(n+1)$-point invariants. Among those m invariants, there are exactly R, say $I_1, \ldots I_R$, such that $\{\Pi^*_{(n+1;1,\ldots,n)}J_1, \ldots, \Pi^*_{(n+1;1,\ldots,n)}J_N, I_1, \ldots, I_R\}$ are functionally independent on an open subset of U_{n+1}.

When $n+1 < n_0$, then a local moving frame doesn't exist in any neighborhood of $z^{(n+1)}$. However, following the partial moving frame normalization method described in [8], we can obtain a map $\tilde{\rho}(p_1, \ldots, p_{n+1})$ such that if we set $g = \tilde{\rho}(p_1, \ldots, p_{n+1})$ in the equation $\bar{p}_{n+1} = g \cdot p_{n+1}$ then we get

$$\bar{p}_{n+1}|_{(g=\tilde{\rho}(p_1,\ldots,p_n))} = (I_1, \ldots, I_m),$$

a vector made of m $(n+1)$-point invariants containing the R invariants we want. □

Corollary 7.10. *There do not exist m n-point joint invariants which are n^{th} point reductive with $n \leq n_o$.*

Theorem 7.11. *Let $n \geq n_0$. If G acts on on open set $U_{n+1} \subset M^{\times(n+1)}$ regularly, then in a neighborhood of any point $z^{(n+1)} \in U_{n+1}$, there exist m $(n+1)$-point joint invariants which are $(n+1)^{st}$ point reductive. These invariants can be obtained via the moving frame method.*

Proof. Obtain $R = m$ $(n+1)$-point functionally independent invariants $\{I_1, \ldots, I_m\}$ as described in the proof of Lemma 7.9. We claim that, on an open subset of U_{n+1}, we can express p_{n+1} as a function

$$p_{n+1} = f(p_1, \ldots, p_n, I_1(p_1, \ldots, p_{n+1}), \ldots, I_m(p_1, \ldots, p_{n+1})).$$

This is because if that were not the case, then the Jacobian matrix

$$\frac{\partial(I_1, \ldots, I_m)}{\partial(p_1, \ldots, p_{n+1})}$$

would contain a sub-matrix

$$\frac{\partial(I_1, \ldots, I_m)}{\partial(p_{n+1})}$$

with rank strictly smaller than m on any open subset of U_{n+1}, which would contradict the fact that, since $n \geq n_0$, the invariants $\{I_1, \ldots, I_m\}$ are (by construction) functionally independent of the pull back by $\Pi_{(n+1;1,\ldots,n)}$ of invariants defined on $\Pi_{(n+1;1,\ldots,n)}U_{n+1}$. □

Let $n^\star$ be the minimum integer n such that G acts on $M^{\times(n)}$ with maximal orbit dimension $s_n < nm$. In other words, $n^\star$ is the minimum n for which non-trivial n-point joint invariants exist. Observe that $n_0 \geq n^\star - 1$.

Lemma 7.12. *Let $J_1, \ldots J_N$ be functionally independent invariants defined on $U_{n^\star} \subset M^{\times(n^\star)}$. The invariants*

$$\begin{array}{ccc} \Pi^\star_{(n^\star+1;1,\ldots,n^\star)}J_1, & \ldots, & \Pi^\star_{(n^\star+1;1,\ldots,n^\star)}J_N, \\ \Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_1, & \ldots, & \Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_N \end{array}$$

are functionally independent on an open subset of $U_{n^\star}|^{n^\star+1}$.

Proof. Observe that $N \leq m$. The Jacobian matrix

$$\frac{\partial(\Pi^\star_{(n^\star+1;1,\ldots,n^\star)}J_1, \ldots, \Pi^\star_{(n^\star+1;1,\ldots,n^\star)}J_N, \Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_1, \ldots, \Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_N)}{\partial(p_1, \ldots, p_{n^\star+1})}$$
$$= \begin{pmatrix} \mathbf{A}(p_1, \ldots, p_{n^\star}), \mathbf{0}_{1\times m} \\ \mathbf{0}_{1\times m}, \mathbf{A}(p_2, \ldots, p_{n^\star+1}) \end{pmatrix},$$

where $\mathbf{A} : M^{\times(n)} \to \mathbb{R}^{N\times nm}$ is an $N \times nm$ matrix. The rank of $\mathbf{A}(p_1, \ldots, p_{n^\star})$ is equal to N at any point $(p_1, \ldots, p_{n^\star})$ of an open and dense subset of $U_{n^\star}$. Since there are no $(n^\star - 1)$-point joint invariants, the sub-matrix

$$\frac{\partial(\Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_1, \ldots, \Pi^\star_{(n^\star+1;2,\ldots,n^\star+1)}J_N)}{\partial p_{n^\star+1}}$$

also has rank N at any point $(p_1, \ldots, p_{n^\star+1})$ of an open and dense subset of $U_{n^\star}|^{n^\star+1}$, thus proving the fact that the Jacobian matrix has maximal rank $2N$ on an open and dense subset of $U_{n^\star}|^{n^\star+1}$. □

Let $N_i = (n^\star + i)m - s_{n^\star+i}$ be the number of fundamental invariants of the action of G on $M^{\times(n^\star+i)}$. In particular, $N_{-2} = N_{-1} = 0$. We can actually refine the previous lemma.

Lemma 7.13. *Let $n \geq n^\star$ and let $\{J_1, \dots J_N\}$ be a complete set of functionally independent invariants defined on $U_n \subset M^{\times(n)}$. Then there exist exactly $(N_{n-n^\star} - N_{n-n^\star-1})$ invariants among $\{\Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_1, \dots, \Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_N\}$, say the first $N_{n-n^\star} - N_{n-n^\star-1}$ ones*

$$\Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_1, \dots, \Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_{N_{n-n^\star} - N_{n-n^\star-1}},$$

such that the invariants

$$\begin{array}{ccc} \Pi^\star_{(n^\star+1;1,\dots,n^\star)} J_1, & \dots, & \Pi^\star_{(n^\star+1;1,\dots,n^\star)} J_N, \\ \Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_1, & \dots, & \Pi^\star_{(n^\star+1;2,\dots,n^\star+1)} J_{N_{n-n^\star} - N_{n-n^\star-1}} \end{array}$$

are functionally independent on an open subset of $U_n|^{n+1}$.

Proof. This follows from the fact that the rank of the sub-matrix

$$\frac{\partial(J_1, \dots, J_N)}{\partial p_n}$$

is equal to $(N_{n-n^\star} - N_{n-n^\star-1})$ on an open and dense subset of $U_n|^{n+1}$. □

As a corollary, we have the following.

Lemma 7.14. *Let $n \geq n^\star$ and let $J_1, \dots J_N$ be a complete set of functionally independent invariants defined on $U_n \subset M^{\times(n)}$. Define $I_1, \dots, I_R$ with $R = N_{n-n^\star+1} - N_{n-n^\star}$ as in Lemma 7.9. There exist*

$$L = R - (N_{n-n^\star} - N_{n-n^\star-1}) = N_{n-n^\star+1} - 2N_{n-n^\star} + N_{n-n^\star-1} \geq 0$$

invariants among $\{I_1, \dots, I_R\}$, say $I_1 \dots, I_L$, such that the invariants

$$\begin{array}{ccc} \Pi^*_{(n+1;1,\dots,n)} J_1, & \dots, & \Pi^*_{(n+1;1,\dots,n)} J_N, \\ \Pi^*_{(n+1;2,\dots,n+1)} J_1, & \dots, & \Pi^*_{(n+1;2,\dots,n+1)} J_N \\ I_1, & \dots, & I_L \end{array}$$

contain a complete fundamental set of invariants on an open subset of $U_n|^{n+1}$. More precisely, one can find R invariants among

$$\left\{ \Pi^*_{(n+1;2,\dots,n+1)} J_1, \dots, \Pi^*_{(n+1;2,\dots,n+1)} J_N \right\},$$

say the first R ones,

$$\Pi^*_{(n+1;2,\dots,n+1)} J_1, \dots, \Pi^*_{(n+1;2,\dots,n+1)} J_R,$$

such that

$$\begin{array}{ccc} \Pi^*_{(n+1;1,\dots,n)} J_1, & \dots, & \Pi^*_{(n+1;1,\dots,n)} J_N, \\ \Pi^*_{(n+1;2,\dots,n+1)} J_1, & \dots, & \Pi^*_{(n+1;2,\dots,n+1)} J_R, \\ I_1, & \dots, & I_L \end{array}$$

is a complete fundamental set of invariants on an open subset of $U_n|^{n+1}$.

The proofs of the next three theorems constitute the core of this paper as they contain the algorithm for the construction of recorders. Theorem 7.15 is a warm up. It concerns a very common case for which the construction requires very few steps. We deal with the general case in a similar manner. For clarity, we present the general construction in two parts corresponding to Theorem 7.16 and Theorem 7.17 respectively. The examples of the special Euclidean and full Euclidean Lie groups already discussed illustrate this construction.

Theorem 7.15 (Case $n_0 = n^\star$ or $n_0 = n^\star - 1$)**.** *Assume that G acts transitively on an open set $U_{n_0-1} \subset M^{\times(n_0-1)}$ and locally freely and regularly on $U_{n_0-1}|^{n_0}$. There exists an open subset of $U_{n_0-1}|^{n_0+1}$ and m invariants $I_1, \ldots, I_m$ which are recorders on this open subset. These invariants can be obtained via the moving frame normalization method.*

Proof. By transitivity of the action of G on U_{n_0-1}, there are no invariants on any subset of U_{n_0-1}. By local freeness of the action of G on $U_{n_0-1}|^{n_0}$, we can use the moving frame normalization method to obtain a complete set of $n_0 m - \delta \geq 0$ invariants defined in a neighborhood of any given point of $U_{n_0-1}|^{n_0}$. To do so, we begin by normalizing the equations $\{g \cdot p_i\}_{i=1}^{n_0}$, for $g \in G$ and $(p_1, \ldots, p_{n_0}) \in U_{n_0-1}|^{n_0}$ as described in [8]. More precisely, we set δ of the equations equal to constants, solve them for $g = (g_1, \ldots, g_\delta) = \rho(p_1, \ldots, p_{n_0})$ and evaluate the remaining equations at $g = \rho(p_1, \ldots, p_{n_0})$. Doing so, we obtain $n_0 m - \delta$ functionally independent invariants $J_1^0, \ldots, J_{n_0 m-\delta}^0$ defined on an open subset of $U_{n_0-1}|^{n_0}$. Observe that $n_0 m - \delta < m$ by Lemma 7.9. We set

$$
\begin{aligned}
I_1 &= \Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_1^0, \\
I_2 &= \Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_2^0, \\
&\vdots \\
I_{n_0 m-\delta} &= \Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_{n_0 m-\delta}^0,
\end{aligned}
$$

thus guaranteeing property $(\star)$ whenever $k > n_0$, for then the set

$$\{I_{1,r}, \quad \ldots, \quad I_{n_0 m-r,r}\}_{r=1}^k,$$

contains a set of invariants depending solely on the first n_0 of the k points which, when restricted to $M^{\times(n_0)}$, explicitly contains a complete fundamental set of invariants on an open subset of $U_{n_0-1}|^{n_0}$. We then evaluate the m functions given by the components of $g \cdot p_{n_0+1}$ at $g = \rho(p_1, \ldots, p_{n_0})$ to obtain m invariants $J_1^1, \ldots, J_m^1$ defined on an open subset of $U_{n_0-1}|^{n_0+1}$. The invariants

$$
\begin{array}{ccc}
\Pi^*_{(n_0+1;1,\ldots,n_0)} J_1^0, & \ldots, & \Pi^*_{(n_0+1;1,\ldots,n_0)} J_{n_0 m-\delta}^0, \\
J_1^1, & \ldots, & J_m^1
\end{array}
$$

form a complete set of fundamental invariants on an open subset of $U_{n_0-1}|^{n_0+1}$. By Lemma 7.12, the invariants

$$
\begin{array}{ccc}
\Pi^*_{(n_0+1;1,\ldots,n_0)} J_1^0, & \ldots, & \Pi^*_{(n_0+1;1,\ldots,n_0)} J_{n_0 m-\delta}^0, \\
\Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_1^0, & \ldots, & \Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_{n_0 m-\delta}^0
\end{array}
$$

are functionally independent on an open subset of $U_{n_0-1}|^{n_0+1}$. By Lemma 7.14, we can choose $L = m - (n_0 m - \delta) = \delta - (n_0 - 1)m > 0$ invariants among $J_1^1, \ldots, J_m^1$, say $J_1^1, \ldots, J_L^1$, such that the set

$$
\begin{array}{ccc}
\{\Pi^*_{(n_0+1;1,\ldots,n_0)} J_1^0, & \ldots, & \Pi^*_{(n_0+1;1,\ldots,n_0)} J_{n_0 m-\delta}^0, \\
\Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_1^0, & \ldots, & \Pi^*_{(n_0+1;2,\ldots,n_0+1)} J_{n_0 m-\delta}^0, \\
J_1^1, & \ldots, & J_L^1\}
\end{array}
$$

is a complete set of functionally independent invariants on an open subset of $U_{n_0-1}|^{n_0+1}$. We set

$$\begin{aligned} I_{n_0 m-\delta+1} &= J_1^1, \\ I_{n_0 m-\delta+2} &= J_2^1, \\ &\vdots \\ I_m &= J_L^1. \end{aligned}$$

By the same argument as in the proof of Theorem 7.11, we can show that $\{I_1, \ldots, I_m\}$ are $(n_0+1)^{st}$ point reductive on an open subset of $U_{n_0-1}|^{n_0+1}$. □

Observe that $(n-1)$-point projectabilitily holds for any set of n-point invariants with $n < n^\star$. (All invariants are constants in that case.) The next theorem takes care of the remaining cases, $n \geq n^\star$. We are thus guaranteed that, for any $n \in \mathbb{N}$, we can obtain a set of cardinality at most m containing n-point joint invariants which are $(n-1)$-point projectable, provided that some slight regularity conditions hold.

Theorem 7.16. *Let $d \in \{0, 1, 2, \ldots\}$, let $k \geq n^\star + d$ and let $U_{n^\star} \subset M^{\times(n^\star)}$. If G acts regularly on $U_{n^\star}|^{n^\star+i}$, for $i = 0, 1, 2, \ldots, d$, then there exist $\bar{m} \leq m$ invariants $I_1, \ldots, I_{\bar{m}}$ defined on $U_{n^\star}|^{n^\star+d}$ such that the set*

$$\{I_{1,r}, \ldots, I_{\bar{m},r}\}_{r=1}^k,$$

contains a subset of invariants which only depend on the first $n^\star + d - 1$ of the k points and which, after restriction to the first $n^\star + d - 1$ factors, form a complete fundamental set of invariants on an open subset of $U_{n^\star}|^{n^\star+d-1}$. These can be obtained via the (partial) moving frame normalization method.

Proof. We begin by normalizing the equations $\{g \cdot p_i\}_{i=1}^{n^\star}$, for $g \in G$ and $p_1, \ldots, p_{n^\star} \in M$ as described in [8]. More precisely, we set $s_{n^\star}$ of the group transformation equations equal to constants, solve these equations for $s_{n^\star}$ components of $g = (g_1, \ldots, g_\delta)$, say $(g_1, \ldots, g_{s_{n^\star}}) = \rho_0(p_1, \ldots, p_{n^\star})$, and evaluate the remaining equations at $(g_1, \ldots, g_{s_{n^\star}}) = \rho_0(p_1, \ldots, p_{n^\star})$. Doing so, we obtain N_0 functionally independent invariants $J_1^0, \ldots, J_{N_0}^0$ defined on an open subset of $U_{n^\star}$. We set

$$\begin{aligned} I_1 &= \Pi^*_{(n^\star+d;d+1,\ldots,d+n^\star)} J_1^0, \\ I_2 &= \Pi^*_{(n^\star+d;d+1,\ldots,d+n^\star)} J_2^0, \\ &\vdots \\ I_{N_0} &= \Pi^*_{(n^\star+d;d+1,\ldots,d+n^\star)} J_{N_0}^0. \end{aligned}$$

(We could also pull back with $\Pi_{(n^\star+d;r+1,\ldots,r+n^\star)}$ for any other $r \in \{1, \ldots, d+1\}$. However, the choice of $r = d+1$ insures that all invariants $I_1, \ldots, I_{N_0}$ explicitly depend on the last point $p_{n^\star+d} \in \Pi_{(n^\star+d;n^\star+d)} M^{\times(n^\star+d)}$, a condition that is necessary in order to obtain $(\star\star)$ later on.)

We then normalize the equation $g \cdot p_{n^\star+1}$ to obtain m invariants $J_1^1, \ldots, J_m^1$ defined on an open subset of $U_{n^\star}|^{n^\star+1}$. More precisely, we set $s_{n^\star+1} - s_{n^\star}$ equations equal to constants and solve for $s_{n^\star+1} - s_{n^\star}$ of the remaining components of $g = (g_1, \ldots, g_\delta)$, say $(g_{s_{n^\star}+1}, \ldots, g_{s_{n^\star+1}-s_{n^\star}}) = \rho_1(p_1, \ldots, p_{n^\star+1})$. The m components of $g \cdot p_{n^\star+1}$ evaluated at $(g_{s_{n^\star}+1}, \ldots, g_{s_{n^\star+1}-s_{n^\star}}) = \rho_1(p_1, \ldots, p_{n^\star+1})$ are the m

invariants ($s_{n^\star+1} - s_{n^\star}$ of which are, of course, constant). By Lemma 7.9, among these m invariants there are exactly $R_1 = N_1 - N_0 \geq 0$, say $J_1^1, \ldots, J_{R_1}^1$, such that

$$\{\Pi^*_{(n^\star+1;1,\ldots,n^\star)} J_1^0, \quad \ldots, \quad \Pi^*_{(n^\star+1;1,\ldots,n^\star)} J_{N_0}^0, J_1^1, \quad \ldots \quad, J_{R_1}^1\}$$

are functionally independent. By Lemma 7.14, there exist exactly $N_1 - 2N_0 \geq 0$ invariants among $\{J_1^1, \ldots, J_{R_1}^1\}$, say $J_1^1, \ldots, J_{N_1-2N_0}^1$, such that

$$\begin{array}{rcl} \Pi^*_{(n^\star+1;1,\ldots,n^\star)} J_1^0, & \ldots, & \Pi^*_{(n^\star+1;1,\ldots,n^\star)} J_{N_0}^0, \\ \Pi^*_{(n^\star+1;2,\ldots,n^\star+1)} J_1^0, & \ldots, & \Pi^*_{(n^\star+1;2,\ldots,n^\star+1)} J_{N_0}^0, \\ J_1^1, & \ldots, & J_{N_1-2N_0}^1 \end{array}$$

contains a complete fundamental set of invariants on an open subset of $U_{n^\star}|^{n^\star+1}$.

We set

$$\begin{array}{rcl} I_{N_0+1} & = & \Pi^*_{(n^\star+d;d,\ldots,n^\star+d)} J_1^1, \\ I_{N_0+2} & = & \Pi^*_{(n^\star+d;d,\ldots,n^\star+d)} J_2^1, \\ & \vdots & \\ I_{N_0+N_1-2N_0} & = & \Pi^*_{(n^\star+d;d,\ldots,n^\star+d)} J_{N_1-2N_0}^1. \end{array}$$

(Again, we make sure that all I_i's explicitly depend on the last point $p_{n^\star+d} \in \Pi_{(n^\star+d;n^\star+d)} M^{\times(n^\star+d)}$).

We have now defined a total of $N_1 - N_0$ of the I_i's. Similarly, if we normalize the equation $g \cdot p_{n^\star+2}$ We obtain m invariants out of which $R_2 = N_2 - N_1$, say $\{J_1^2, \ldots, J_{R_2}^2\}$, are such that

$$\begin{array}{rcl} \Pi^*_{(n^\star+2;1,\ldots,n^\star)} J_1^0, & \ldots, & \Pi^*_{(n^\star+2;1,\ldots,n^\star)} J_N^0, \\ \Pi^*_{(n^\star+2;1,\ldots,n^\star+1)} J_1^1, & \ldots, & \Pi^*_{(n^\star+2;1,\ldots,n^\star+1)} J_{R_1}^0, \\ J_1^2, & \ldots, & J_{R_2}^2 \end{array}$$

are functionally independent. By Lemma 7.14, there exist exactly $N_2 - 2N_1 + N_0 > 0$ invariants among $\{J_1^2, \ldots, J_{R_2}^2\}$, say $J_1^2, \ldots, J_{N_2-2N_1+N_0}^2$, such that

$$\begin{array}{rcl} \Pi^*_{(n^\star+2;1,\ldots,n^\star)} J_1^0, & \ldots, & \Pi^*_{(n^\star+2;1,\ldots,n^\star)} J_N^0, \\ \Pi^*_{(n^\star+2;2,\ldots,n^\star+1)} J_1^0, & \ldots, & \Pi^*_{(n^\star+2;2,\ldots,n^\star+1)} J_N^0, \\ \Pi^*_{(n^\star+2;3,\ldots,n^\star+2)} J_1^0, & \ldots, & \Pi^*_{(n^\star+2;3,\ldots,n^\star+2)} J_N^0, \\ \Pi^*_{(n^\star+2;1,\ldots,n^\star+1)} J_1^1, & \ldots, & \Pi^*_{(n^\star+2;1,\ldots,n^\star+1)} J_{R_1}^1, \\ \Pi^*_{(n^\star+2;2,\ldots,n^\star+2)} J_1^1, & \ldots, & \Pi^*_{(n^\star+2;2,\ldots,n^\star+2)} J_{R_1}^1, \\ J_1^2, & \ldots, & J_{N_2-2N_1+N_0}^2 \end{array}$$

contains a complete fundamental set of invariants on an open subset of $U_{n^\star}|^{n^\star+2}$. More precisely, the set

$$\begin{array}{rcl}
\{\Pi^*_{(n^\star+2;1,\dots,n^\star)}J^0_1, & \dots, & \Pi^*_{(n^\star+2;1,\dots,n^\star)}J^0_N,\\
\Pi^*_{(n^\star+2;2,\dots,n^\star+1)}J^0_1, & \dots, & \Pi^*_{(n^\star+2;2,\dots,n^\star+1)}J^0_N,\\
\Pi^*_{(n^\star+2;3,\dots,n^\star+2)}J^0_1, & \dots, & \Pi^*_{(n^\star+2;3,\dots,n^\star+2)}J^0_N,\\
\Pi^*_{(n^\star+2;1,\dots,n^\star+1)}J^1_1, & \dots, & \Pi^*_{(n^\star+2;1,\dots,n^\star+1)}J^1_{N_1-2N_0},\\
\Pi^*_{(n^\star+2;2,\dots,n^\star+2)}J^1_1, & \dots, & \Pi^*_{(n^\star+2;2,\dots,n^\star+2)}J^1_{N_1-2N_0},\\
J^2_1, & \dots, & J^2_{N_2-2N_1+N_0}
\end{array}$$

is a complete fundamental set of invariants on an open subset of $U_{n^\star}|^{n^\star+2}$.

We set

$$\begin{array}{rcl}
I_{N_1-N_0+1} & = & \Pi^*_{(n^\star+d;d-1,\dots,n^\star+d)}J^2_1,\\
I_{N_1-N_0+2} & = & \Pi^*_{(n^\star+d;d-1,\dots,n^\star+d)}J^2_2,\\
 & \vdots & \\
I_{N_1-N_0+N_2-2N_1+N_0} & = & \Pi^*_{(n^\star+d;d-1,\dots,n^\star+d)}J^2_{N_2-2N_1-N_0}.
\end{array}$$

We have now defined N_2-N_1 of the I_i's. Following this procedure d times, we obtain $(N_{d-1}-N_{d-2})$ functionally independent invariants

$$\{I_1, \quad \dots \quad, I_{N_d-N_{d-1}}\},$$

defined on some open subset of $U_{n^\star}|^{n^\star+d}$. We claim that $\{I_{1,r},\dots,I_{N_d-N_{d-1},r}\}_{r=1}^k$ contains a set of N_{d-1} invariants depending only on the first $n^\star+d-1$ of the k points and which are functionally independent on $\Pi_{(n^\star+d;1,\dots,n^\star+d-1)}\,U_{n^\star}|^{n^\star+d}$. Indeed, by construction, the set

$$\begin{array}{rl}
\Omega = & \{J^{d-1}_{1,r}, \quad \dots \quad, J^{d-1}_{N_{d-1}-2N_{d-2}+N_{d-3},r}\}\\
\cup & \{J^{d-2}_{1,r}, \quad \dots \quad, J^{d-2}_{N_{d-2}-2N_{d-3}+N_{d-4},r}\}_{r=1}^2\\
 & \vdots\\
\cup & \{J^0_{1,r}, \quad \dots \quad, J^0_{N_0-2N_{-1}+N_{-2},r}\}_{r=1}^d,
\end{array}$$

which is a subset of $\{I_{1,r},\dots,I_{N_{d-1}-N_{d-2},r}\}_{r=1}^k$, contains exactly N_{d-1} functionally independent invariants.

Observe that

$$\begin{array}{rcl}
N_{d-1}-N_{d-2} & = & (n^\star+d-1)m-s_{d-1}-((n^\star+d-2)m-s_{d-2})\\
 & = & m-s_{d-1}+s_{d-2}\\
 & \leq & m,
\end{array}$$

which completes the proof. □

Note that, in the previous proof, the invariants $I_1,\dots,I_{\bar m}$ could have been defined on $U_{n^\star}|^{n^\star+d-1}$ rather than on the bigger set $U_{n^\star}|^{n^\star+d}$ since they do not actually depend at all on the first of the $n^\star+d$ points. But our goal was to construct a set of $(n^\star+d-1)$-point projectable invariants to which we could simply add a few more invariants in order to obtain a recording set. Defining $\{I_1,\dots,I_{\bar m}\}$ on $U_{n^\star}|^{n^\star+d}$

allows us to do so directly, without having to "shift" the domain of definition of the invariants $\{I_1, \ldots, I_{\bar{m}}\}$ we already have. This thus simplifies the proof of the next theorem, which builds on the results of the previous theorem to prove the existence of recording n-point invariants for any $n \geq n_0 + 1$.

Theorem 7.17 (General case)**.** *Let $d \in \{0, 1, 2, \ldots\}$ and $U_{n^\star} \subset M^{\times (n^\star)}$ Assume that G acts regularly on $U_{n^\star}|^{n^\star+i} \subset M^{\times(n^\star+i)}$, for $i = 0, 1, 2, \ldots, d$. If $n^\star + d > n_0$ then, on an open subset of $U_{n^\star}|^{n^\star+d}$, there exists m $(n^\star + d)$-point joint invariants $I_1, \ldots, I_m$ which are recorders. These invariants can be obtained via the (partial) moving frame normalization method.*

Proof. We start with constructing $\{I_1, \ldots, I_{N_{d-1}-N_{d-2}}\}$ as in the proof of Theorem 7.16. Then we normalize the equation $g \cdot p_{n^\star+d}$, for $g \in G$ and $p_{n^\star+d} \in M$, and repeat the exact same procedure as in the previous theorem to obtain a total of $N_d - N_{d-1}$ invariants $I_1, \ldots, I_{N_d - N_{d-1}}$. More precisely, we choose some $N_d - 2N_{d-1} + N_{d-2}$ invariants among $J^d_1, \ldots, J^d_{N_d - N_{d-1}}$, say $J^d_1, \ldots, J^d_{N_d - 2N_{d-1} + N_{d-2}}$, such that for the complete set of invariants $\{J^{d-1}_{1,1}, \quad \ldots \quad, J^{d-1}_{N_{d-1},1}\}$ on some open subset of $U_{n^\star}|^{n^\star+d-1}$, we have that

$$
\begin{array}{rcl}
\Pi^*_{(n^\star+d;1,\ldots,n^\star+d-1)} J^{d-1}_1, & \ldots, & \Pi^*_{(n^\star+d;1,\ldots,n^\star+d-1)} J^{d-1}_{N_{d-1}} \\
\Pi^*_{(n^\star+d;2,\ldots,n^\star+d)} J^{d-1}_1, & \ldots, & \Pi^*_{(n^\star+d;2,\ldots,n^\star+d)} J^{d-1}_{N_{d-1}} \\
J^d_1, & \ldots, & J^d_{N_d - 2N_{d-1} + N_{d-2}}
\end{array}
$$

are functionally independent. We set

$$
\begin{array}{rcl}
I_{N_{d-1}-N_{d-2}+1} & = & J^d_1 \\
I_{N_{d-1}-N_{d-2}+2} & = & J^d_2 \\
& \vdots & \\
I_{N_{d-1}-N_{d-2}+N_d-2N_{d-1}+N_{d-2}} & = & J^d_{N_d-2N_{d-1}+N_{d-2}},
\end{array}
$$

this obtaining $N_d - N_{d-1}$ invariants $\{I_1, \ldots, I_{N_d - N_{d-1}}\}$.

Since $\{I_1, \ldots, I_{\bar{m}}\} \subset \{I_{1,r}, \ldots, I_{N_d - N_{d-1}, r}\}$, the set

$$
\{I_{1,r}, \quad \ldots, \quad I_{N_d - N_{d-1}, r}\}_{r=1}^k
$$

explicitly contains (after restriction to the first $n^\star + d$ factors) a complete fundamental set of invariants on $\Pi_{(n^\star+d;1,\ldots,n^\star+d-1)}\, U_{n^\star}|^{n^\star+d}$. Moreover, since $n^\star + d > n_0$, we have $N_d - N_{d-1} = m$.

By the same argument as in the proof of Theorem 7.11, we can show that $\{I_1, \ldots, I_m\}$ are $(n^\star + d)^{th}$ point reductive. □

An immediate corollary of Corollary 7.10, we also have

Theorem 7.18. *If $n < n_0$, then for any open set $U_{n+1} \subset M^{\times(n+1)}$, there do not exist m $(n + 1)$-point joint invariants which are recorders on U_{n+1}.*

For the purpose of partial recognition, it is certainly better to use invariants depending on as few points as possible. One should thus try to build an $(n_0 + 1)$-point signature, which is the optimal number for any Lie group. However, taking more points than the minimum sometimes allows for $(\star\star)$ to be true on a bigger domain, making the detection algorithm applicable in more cases, as illustrated by the example of the Euclidean group acting the plane.

The following sections contain explicit JIS curves with examples for some less intuitive Lie group actions, namely the the equi-affine group $SA(2)$, the skewed affine group $SKA(2)$ and the similarity group $SIM(2)$ acting on the plane as well as $SL(2)$ acting on the Poincaré half-plane.

8. $SA(2)$ SYMMETRY DETECTION USING $SAJIS$ CURVES

The equi-affine group $SA(2)$ is the group of area and orientation preserving transformations in the plane. For $p \in \mathbb{R}^2$, the group transformation can be written as

$$g \cdot p = Ap + v,$$

with $A \in SL(2)$ and $v \in \mathbb{R}^2$. The Cartesian group action becomes free on some open set as soon as $SA(2)$ acts on three copies of the plane. It is also regular on $\{(p_1,\ldots,p_n) \in (\mathbb{R}^2)^{\times(n)} | \; p_1,\ldots,p_n \text{ are distinct and not colinear }\}$, for all positive integers n. The corresponding maximal orbit dimensions are

$$\begin{aligned} 2 \text{ when } n &= 1 \\ 4 \text{ when } n &= 2 \\ 5 \text{ when } n &\geq 3 \end{aligned}$$

Therefore, there are no invariants of the Cartesian action on one or two copies of the plane, while there is one fundamental invariant on three copies, and three fundamental invariants on four copies of the plane. Since $n_0 = 3$, we will try to build an $n_0 + 1 = 4$-point equi-affine joint invariant signature ($SAJIS$).

Let $\Delta_{ijk} = \frac{1}{2}(p_j - p_i) \wedge (p_j - p_k)$ be the signed area of the triangle spanned by the vectors $p_j - p_i$ and $p_j - p_k$, for $p_i, p_j, p_k \in \mathbb{R}^2$. The following are the results obtained directly from the moving frame normalization method [9].

Theorem 8.1. *For $SA(2)$ acting on $M = \mathbb{R}^2$, we have the following.*

(1) *There are no one-point joint invariants.*
(2) *There are no two-point joint invariants.*
(3) *There is one fundamental three-point joint invariants*

$$J_1^0(p_1,p_2,p_3) : (\mathbb{R}^2)^{\times(3)} \to \mathbb{R},$$

which can be taken as $\quad J_1^0(p_1,p_2,p_3) = 2\Delta_{123}.$

(4) *There are three fundamental four-point joint invariants*

$$J_1^1(p_1,p_2,p_3,p_4), \quad J_2^1(p_1,p_2,p_3,p_4), \quad J_3^1(p_1,p_2,p_3,p_4) : (\mathbb{R}^2)^{\times(4)} \to \mathbb{R},$$

which can be taken as

$$\begin{aligned} J_1^1(p_1,p_2,p_3,p_4) &= 2\Delta_{123}, \\ J_2^1(p_1,p_2,p_3,p_4) &= -\frac{\Delta_{134}}{\Delta_{123}}, \\ J_3^1(p_1,p_2,p_3,p_4) &= 2\Delta_{124}. \end{aligned}$$

According to the construction described in the proof of Theorem 7.17, we take $I_1 = J_{1,2}^0 = 2\Delta_{234}$. Observe that $J_{1,1}^0 = J_1^1$. So for I_2, we are free to take any invariant among the J_i^1's *except* J_1^1, as long as I_2 and $\{J_{1,r}^0\}_{r=1}^2$ are functionally independent. In fact, we could take $I_2 = -\frac{\Delta_{134}}{\Delta_{123}}$ or $I_2 = 2\Delta_{124}$. For simplicity, we get rid of the constants and choose to take

$$I_1(p_1,p_2,p_3,p_4) = \Delta_{234} \quad \text{and} \quad I_2(p_1,p_2,p_3,p_4) = \Delta_{124}.$$

By construction, $\{I_1, I_2\}$ are three-point projectable ($\star$) and fourth point reductive ($\star\star$) on some neighborhood of any point $z^{(4)}$ such that

$$z^{(4)} \in \{(p_1,p_2,p_3,p_4) \in (\mathbb{R}^2)^{\times(4)} | \quad p_1,p_2,p_3,p_4 \text{ are distinct and not colinear } \}.$$

In fact, ($\star$) holds for all planar polygon assuming all four consecutive vertices are distinct. In order to know exactly where ($\star\star$) holds, we solve the equations

$$I_1(p_1,p_2,p_3,p_4) = c_1, \quad I_2(p_1,p_2,p_3,p_4) = c_2$$

for p_4. Computations show that a unique solution

$$p_4 = f(p_1,p_2,p_3,I_1(p_1,p_2,p_3,p_4),I_2(p_1,p_2,p_3,p_4))$$

exists, provided that p_1, p_2 and p_3 do not lie on a straight line. Since $SA(2)$ acts regularly on $U_3 = \{(p_1,p_2,p_3)|\Delta_{123} \neq 0\}$ and I_1 and I_2 are recorders on U_3, our $SAJIS$ will characterize all planar polygons for which no three consecutive vertices lie on a straight line.

We wrote a MATLAB routine to test our signature on actual polygons. Using the $SAJIS$, we were able to detect equi-affine symmetries on a collection of test polygons. One of our test polygons is shown in Figure 6. It is an example of a polygon with some non-trivial affine symmetry. It was constructed by taking a polygon with a four-fold rotational symmetry and four axes of Euclidean symmetry and by applying a linear transformation $T \in SA(2)\backslash SE(2)$. Therefore, it has a four-fold equi-affine symmetry and four axes of skewed-affine symmetry which are *not* Euclidean symmetries. Indeed the $SEJIS$ and $EJIS$ curves (not shown) confirmed that there is no Euclidean symmetry. On the other hand, for a counterclockwise traveling direction, computations gave the following $SAJIS$.

$$SAJIS = \begin{bmatrix} 1 & -2 & -2 & 1 & -2 & -2 & 1 & -2 & -2 & 1 & -2 & -2 \\ -2 & 1 & -2 & -2 & 1 & -2 & -2 & 1 & -2 & -2 & 1 & -2 \end{bmatrix}$$

As one can see from the repetitions in the $SAJIS$ (winding number equal to four), this figure has a four-fold equi-affine symmetry.

9. $SKA(2)$ SYMMETRY DETECTION USING $SKAJIS$ CURVES

The skewed-affine group $SKA(2)$ is the group of area preserving transformations in the plane. On any $p \in \mathbb{R}^2$, the group action can be written exactly as for the previous group,

$$g \cdot p = Ap + v,$$

where the only difference with the case of $SA(2)$ is that $det(A) = \pm 1$. This theorem can be proved using the moving frame method.

Theorem 9.1. *For $SKA(2)$ acting on $M = \mathbb{R}^2$, we have the following.*

1. *There are no one-point joint invariants.*
2. *There are no two-point joint invariants.*
3. *There is one fundamental three-point joint invariants $J_1^0(p_1,p_2,p_3) : (\mathbb{R}^2)^{\times(3)} \to \mathbb{R}$, which can be taken as $J_1^0(p_1,p_2) = 2|\Delta_{123}|$.*

(4) *There are three fundamental four-point joint invariants*

$$J_1^1(p_1,p_2,p_3,p_4), \quad J_2^1(p_1,p_2,p_3,p_4), \quad J_3^1(p_1,p_2,p_3,p_4) : (\mathbb{R}^2)^{\times(4)} \to \mathbb{R},$$

which can be taken as

$$\begin{aligned} J_1^1(p_1,p_2,p_3,p_4) &= 2|\Delta_{123}|, \\ J_2^1(p_1,p_2,p_3,p_4) &= -\frac{\Delta_{134}}{\Delta_{123}}, \\ J_3^1(p_1,p_2,p_3,p_4) &= 2|\Delta_{124}|. \end{aligned}$$

We can try to build a $n_0+1=4$ point signature with, for example, $I_1=|\Delta_{234}|$ and $I_2=|\Delta_{124}|$. But then property $(\star\star)$ only holds for convex polygons, which is a severe restriction. Inspired by our results with the Euclidean group, we choose to take

$$I_1 = sign(\Delta_{123}\Delta_{234})|\Delta_{234}| \quad \text{and} \quad I_2 = sign(\Delta_{123}\Delta_{124})|\Delta_{124}|.$$

These two invariants satisfy property $(\star)$ because $\Delta_{123}=|I_{2,N}|$. Moreover, computations show that we can solve for

$$p_4 = f(p_1,p_2,p_3,I_1(p_1,\dots,p_4),I_2(p_1,\dots,p_4))$$

provided p_1, p_2 and p_3 do not lie on a straight line. So I_1 and I_2 are recorders on $U_4|^k$, for any $k\geq 4$, where $U_4=\{(p_1,p_2,p_3,p_4)\in M^{\times(4)}|\Delta_{123}\neq 0\}$. Since $SKA(2)$ acts regularly on $\{(p_1,p_2,p_3)|\Delta_{123}\neq 0\}$, I_1 and I_2 can be used to recognize all polygons $P=\langle p_1,\dots,p_k\rangle$ such that $(p_1,\dots,p_k)\in U_4|^k$.

The polygon contained in Figure 6 gives the following $SKAJIS$ for a counter-clockwise orientation,

$$SKAJIS_1 = \begin{bmatrix} -2 & 2 & -1 & -2 & 2 & -1 & -2 & 2 & -1 & -2 & 2 & -1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 \end{bmatrix},$$

and the following $SKAJIS$ for a clockwise direction,

$$SKAJIS_2 = \begin{bmatrix} -2 & 2 & -1 & -2 & 2 & -1 & -2 & 2 & -1 & -2 & 2 & -1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 \end{bmatrix}.$$

As one can see from the symmetries and repetitions in the $SKAJIS$, this figure has a four-fold equi-affine symmetry (winding number equal to four) and four axes of skewed-affine symmetry (shown in Figure 6).

10. $SIM(2)$ SYMMETRY DETECTION USING $SIMJIS$ CURVES

Another important group is the similarity group given by all special Euclidean and scaling transformations of the form

$$g\cdot p = \lambda A p + b$$

with $\lambda\in\mathbb{R}^+$, $A\in SO(2)$ and $b\in\mathbb{R}^2$, for any $p\in\mathbb{R}^2$. Observe that this group acts locally freely and transitively on $\{(p_1,p_2)\in(\mathbb{R}^2)^{\times(2)}|\quad p_1\neq p_2\}$. Using the moving frame method and following the construction described in this paper, we chose to take the following two invariants

$$I_1(p_1,p_2,p_3) = \frac{\Delta_{123}}{|p_2-p_1|^2}, \qquad I_1(p_1,p_2,p_3) = \frac{(p_2-p_1)\cdot(p_3-p_1)}{|p_2-p_1|^2},$$

which are recorders on $\{(p_1, p_2, p_3) \in (\mathbb{R}^2)^{\times(3)} | \quad p_1, p_2, p_3 \text{ are distinct } \}$. In order to test the corresponding similarity joint invariant signature ($SIMJIS$), we computed the $SIMJIS$ associated to a collection of polygons with some rotational symmetry and checked that the signature did illustrate the symmetry. Observe that a polygon *cannot* have a scaling symmetry. So, of course, only rotational symmetries are indicated by the $SIMJIS$; the scaling part of the similarity group is of interest only when comparing two polygons. An example of two polygons equivalents under a scaling transformation is presented in Figure 7. The associated signature for a counterclockwise orientation

$$SIMJIS = \begin{bmatrix} 0.5 & -2 & 1 & 1 & -1 & 1 & 1 & -0.5 & 2 & 0.5 & -1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

is the same in both cases. For clarity, we did not graph the arrows representing the direction of each segment joining consecutive points of the signature curve.

FIGURE 5

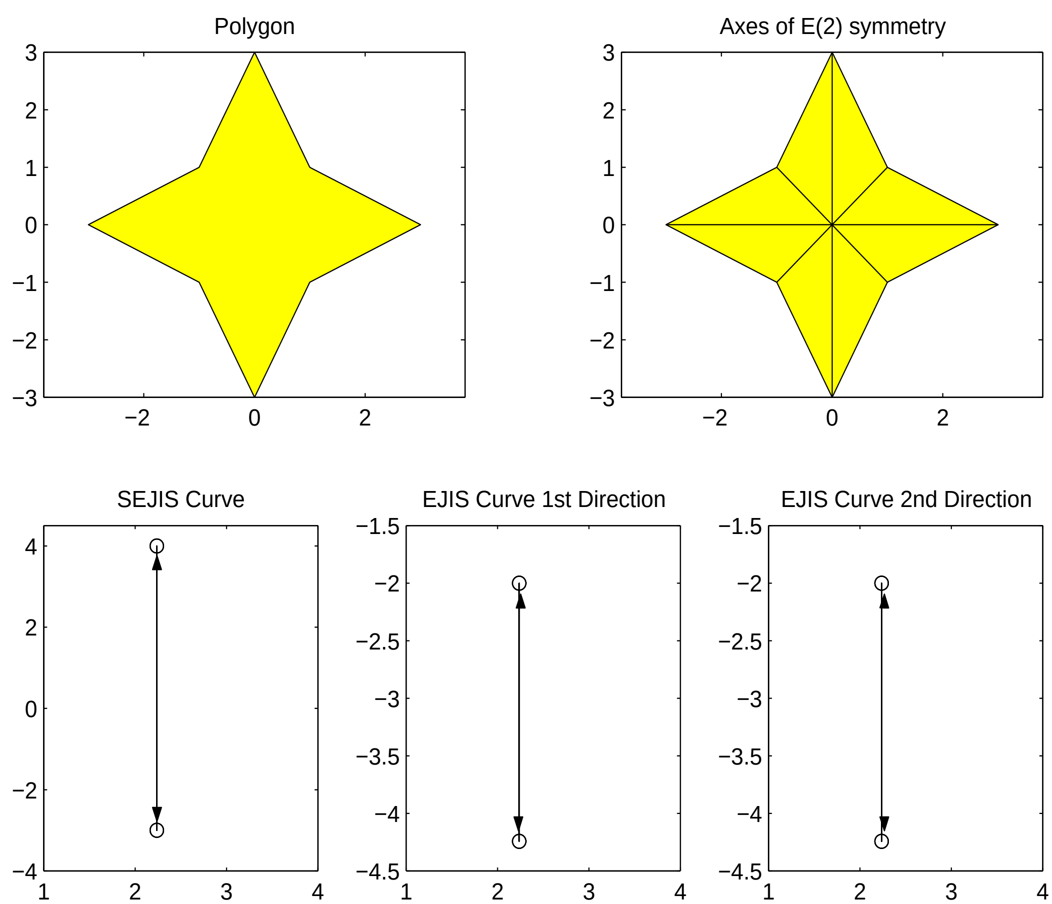
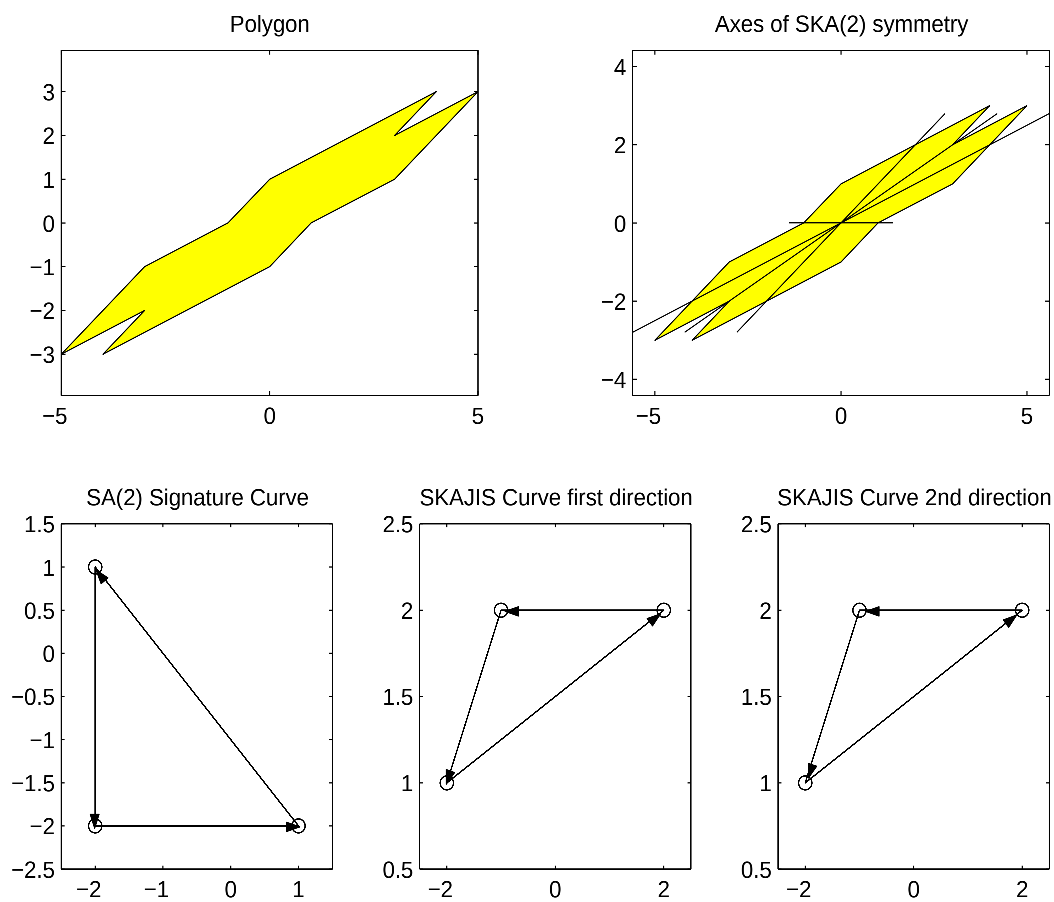


FIGURE 6



11. Polygon recognition in the Poincaré half-plane

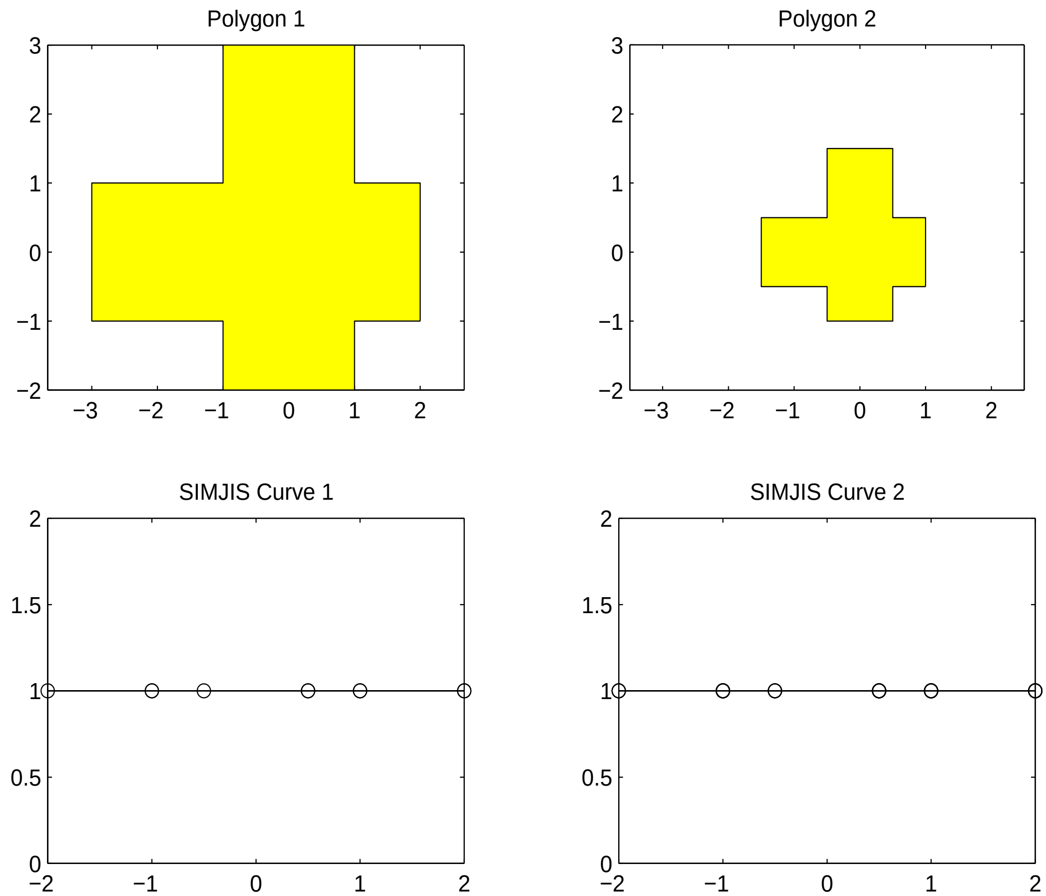
As a final example, let us consider the more challenging case of polygons in the Poincaré half-plane $\mathfrak{H}$ under the action of $SL(2)$. More precisely, let $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ and consider the action given by

$$g \cdot z = \frac{az+b}{cz+d},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$. Note that the Poincaré half-plane is a complete Riemannian manifold so any two points of $\mathfrak{H}$ can be linked by a geodesic. In fact, this geodesic is unique. Moreover, geodesics are preserved under the action of $SL(2)$ so in this case, equivalence of the sequence of vertices implies equivalence of the polygon in the traditional sense (edges mapped to edges).

Since this action is transitive, there are no one-point joint invariants. The prolonged $SL(2)$ action on two copies of the Poincaré half-plane has three-dimensional

FIGURE 7



orbits so there is one fundamental two-point joint invariant and the stabilization order is $n_0 = n^* = 2$.

To obtain such an invariant, we consider two distinct points $z_1, z_2 \in \mathfrak{H}$ and the unique geodesic passing through them. If $\mathrm{Re}(z_1) \neq \mathrm{Re}(z_2)$, the geodesic is a circle with center on the real axis while if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, then the geodesic is a straight line parallel to the imaginary axis. Consider z_1^* and z_2^*, the two real (or infinite) numbers lying on the geodesic passing through z_1 and z_2 labeled in such a way that the orientation of the segment $\overline{z_1 z_2}$ is the same as the orientation of the segment $\overline{z_1^* z_2^*}$. In particular, if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$, then $z_1 = 0$ and $z_2 = \infty$. We use the moving frame normalization method to obtain a fundamental two-point joint invariant J_1^0. We set $g \cdot z_1^* = 0$ and $g \cdot z_2^* = \infty$ and solve for g. The solution is given by

$$b = -a z_1^* \text{ and } d = -c z_2^*. \tag{7}$$

Such a transformation will of course map the real parts of both z_1 and z_2 to zero. We then impose the additional condition that $g \cdot z_1 = i$, which implies that

$$\frac{a}{c} = i\frac{z_1 - z_2^*}{z_1 - z_1^*}. \tag{8}$$

Equations 7 and 8 define a moving frame. Substituting this moving frame for g in the expression for $g \cdot z_2$, we get the invariant $i\frac{(z_1-z_2^*)(z_2-z_1^*)}{(z_1-z_1^*)(z_2-z_2^*)}$. For simplicity we take $J_1^0 = \frac{(z_1-z_2^*)(z_2-z_1^*)}{(z_1-z_1^*)(z_2-z_2^*)}$ as our fundamental two-point joint invariant. Observe that J_1^0 is a real positive number. In fact, $\log J_1^0$ is the hyperbolic distance between z_1 and z_2.

On three copies of $\mathfrak{H}$, the maximal orbit dimension is also equal to three so there are $6-3=3$ fundamental invariants. We can take J_1^0 as a first fundamental invariant and obtain two other invariants by replacing the moving frame into $g \cdot z_3 = \frac{az_3+b}{cz_3+d}$. The norm and argument (or the real and imaginary part) of the resulting expression $\mathcal{E}(z_1,z_2,z_3) = i\frac{(z_1-z_2^*)(z_3-z_1^*)}{(z_1-z_1^*)(z_3-z_2^*)}$ can be used as our second and third fundamental invariants J_1^1 and J_2^1.

One possible choice for the invariants I_1 and I_2 to be used for parameterizing a signature is

$$I_1 = J_{1,2}^0 = \frac{(z_1 - z_3^\circ)(z_3 - z_1^\circ)}{(z_1 - z_1^\circ)(z_3 - z_3^\circ)}, \tag{9}$$

$$I_2 = \text{ sign } (\text{Im } \mathcal{E}(z_1,z_2,z_3))\, |\mathcal{E}(z_1,z_2,z_3)|, \tag{10}$$

where z_2° and z_3° are the two real (or infinite) numbers lying on the geodesic passing through z_2 and z_3 and labeled with the same orientation.

Property $(\star)$ is guaranteed by the fact that two pairs of points in the Poincaré half-plane can be mapped onto each other by a transformation $g \in SL(2)$ if and only if their hyperbolic distance is the same. To find out where $(\star\star)$ is valid, we need to determine where one can solve for z_3 as a function of z_1, z_2 and $I_1(z_1,z_2,z_3)$, $I_2(z_1,z_2,z_3)$. Observe that, given distinct $z_1, z_2 \in \mathfrak{H}$, we can consider the real numbers z_1^* and z_2^* previously defined and find $g_0 \in SL(2)$ such that $g_0 \cdot (z_1^*, z_2^*, z_1) = (0, \infty, i)$. By invariance,

$$I_2(z_1,z_2,z_3) = I_2(g_0\cdot z_1, g_0 \cdot z_2, g_0 \cdot z_3) = -\text{sign } (\text{Re } (g_0 \cdot z_3))|g_0 \cdot z_3|.$$

So $I_2(z_1,z_2,z_3)$ prescribes the quadrant in which $g_0 \cdot z_3$ lies and the Euclidean distance from $g_0 \cdot z_3$ to the origin, provided Re $(g_0 \cdot z_3) \neq 0$. On the other hand $I_1(z_1,z_2,z_3)$ prescribes the hyperbolic distance between $g_0 \cdot z_2$ and $g_0 \cdot z_3$. This means that $g_0 \cdot z_3$ is uniquely prescribed by z_1, z_2, the value of $I_1(z_1,z_2,z_3)$ and the value of $I_2(z_1,z_2,z_3)$, provided that Re $(g_0 \cdot z_3) \neq 0$. Therefore z_3 is uniquely prescribed by $I_1(z_1,z_2,z_3)$, $I_2(z_1,z_2,z_3)$ and distinct z_1, z_2, provided it does not lie on the geodesic passing through z_1 and z_2.

We conclude that the invariants I_1 and I_2 as given by Equations 9 and 10 can be used for recognition up to $SL(2)$ of polygons for which no three consecutive vertices lie on a geodesic.

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REFERENCES

[1] M. Boutin. On orbit dimensions under a simultaneous lie group action on n copies of a manifold. *J. Lie Theory*, 12:191–203, 2002.

[2] M. Boutin. Joint invariant signatures for curve recognition. In *Interaction of Inverse Problems and Imaga analysis*, Contemporary Mathematics. AMS, to appear.

[3] É. Cartan. *La méthode du repère mobile, la théorie des groupes continus et les espaces généralisés.* Exposés de géométrie No.5. Hermann, Paris, 1935.

[4] M. Fels and P. J. Olver. Moving coframes. I. a practical algorithm. *Acta Appl. Math.*, 51:161–213, 1998.

[5] M. Fels and P. J. Olver. Moving coframes. II. regularization and theoretical foundations. *Acta Appl. Math.*, 55:127–208, 1999.

[6] A. Helmut and J. Bloemer. *Resemblance and symmetries of geometric patterns, Data structures and efficient algorithms (Berlin, 1991),1–24.* Lectures Notes in Comput. Sci., 594. Springer, Berlin, 1992.

[7] P. J. Olver. *Equivalence, Invariants and Symmetry.* Cambridge University Press, 1995.

[8] P. J. Olver. *Classical Invariant theory.* London Mathematical Society Student Texts. Cambridge University Press, 1999.

[9] P. J. Olver. Joint invariant signatures. *Found. Comput. Math.*, 1:3–67, 2001.

[10] A. V. Tuzikov, K. Kheĭmans, G. L. Margolin, and S. A. Sheĭnin. Similarity measures of convex polygons. *(Russian) Dokl. Nats. Akad. Nauk Belarusi*, 42(2):40–44, 1998.

[11] J. D. Wolter, T. C. Woo, and R. A. Volz. Optimal algorithms for symmetry detection in 2 and 3 dimensions. *Visual Comput.*, 1:37–48, 1985.

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