EE 438 Assignment #4 solutions Spring '01

1. a. Let rectangular window $w_R(n)$ be

$$w_R(n) = \begin{cases} 1, & n = 0, 1, 2, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$DTFT\{w_R(n)\} = W_R(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\frac{\omega N}{2}}(2j)\sin(\frac{\omega N}{2})}{e^{-j\frac{\omega}{2}}(2j)\sin(\frac{\omega}{2})}$$

$$= e^{-j\frac{\omega(N-1)}{2}} \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}$$

Then $x(n) = \cos(\omega_0 n) w_R(n)$, where $\omega_0 = 2\pi k_0/N$

Use modulation property to find $X(e^{j\omega})$:

$$X(e^{j\omega}) = \frac{1}{2} W_R(e^{j(\omega-\omega_0)}) + \frac{1}{2} W_R(e^{j(\omega+\omega_0)})$$

Now $X_k(k) = X(e^{j\omega})|_{\omega_k = 2\pi k/N}$, $k = 0, 1, \ldots, N-1$,

$$X(k) = \frac{1}{2} \frac{\sin[\frac{N}{2}(\frac{2\pi k}{N} - \frac{2\pi k_0}{N})]}{\sin[\frac{1}{2}(\frac{2\pi k}{N} - \frac{2\pi k_0}{N})]} e^{-j\frac{2\pi(k-k_0)}{N}\cdot\frac{(N-1)}{2}}$$

$$+ \frac{1}{2} \frac{\sin[\frac{N}{2}(\frac{2\pi k}{N} + \frac{2\pi k_0}{N})]}{\sin[\frac{1}{2}(\frac{2\pi k}{N} + \frac{2\pi k_0}{N})]} e^{-j\frac{2\pi(k+k_0)}{N}\cdot\frac{(N-1)}{2}}$$

$$= \frac{1}{2} \frac{\sin[\pi(k-k_0)]}{\sin[\frac{\pi}{N}(k-k_0)]} e^{-j\frac{\pi(k-k_0)(N-1)}{N}}$$

$$+ \frac{1}{2} \frac{\sin[\pi(k+k_0)]}{\sin[\frac{\pi}{N}(k+k_0)]} e^{-j\frac{\pi(k+k_0)(N-1)}{N}}$$

$$= \begin{cases} N/2, & k = k_0, N-k_0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\boxed{= \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta(k-(N-k_0))}$$

Comment for part a: No truncation or leakage effects.

b. Now $\omega_0 = 2\pi(k_0+0.5)/N$.

$$X(k) = \frac{1}{2}\,\frac{\sin[\pi(k-k_0-0.5)]}{\sin[\frac{\pi}{N}(k-k_0-0.5)]}\,e^{-j\frac{\pi(k-k_0-0.5)(N-1)}{N}} + \frac{1}{2}\,\frac{\sin[\pi(k+k_0+0.5)]}{\sin[\frac{\pi}{N}(k+k_0+0.5)]}\,e^{-j\frac{\pi(k+k_0+0.5)(N-1)}{N}}$$

Comment: truncation and leakage are apparent

c. $$X(k) = \sum_{n=0}^{N-1} x(n)\,e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N/4-1} e^{-j\frac{2\pi kn}{N}} + \sum_{n=N/2}^{3N/4-1} e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N/4-1} e^{-j\frac{2\pi kn}{N}} + e^{-j\frac{2\pi k(N/2)}{N}} \sum_{n=0}^{N/4-1} e^{-j\frac{2\pi kn}{N}}$$

$$= [1+(-1)^k] \sum_{n=0}^{N/4-1} e^{-j\frac{2\pi kn}{N}}$$

$$= [1+(-1)^k]\,\frac{1-e^{-j\frac{2\pi k(N/4)}{N}}}{1-e^{-j\frac{2\pi k}{N}}}$$

$$= [1+(-1)^k]\,\frac{e^{-j\frac{\pi k}{4}}\sin\frac{\pi k}{4}}{e^{-j\frac{\pi k}{N}}\sin\frac{\pi k}{N}}$$

$$= \begin{cases} N/2, & \text{if } k=0 \\ 2e^{-j\frac{\pi k}{4}}\sin\frac{\pi k}{4} \Big/ e^{-j\frac{\pi k}{N}}\sin\frac{\pi k}{N}, & k \text{ even but } k\neq 0 \\ 0, & \text{otherwise} \end{cases}$$

2. a. $y(n) = \frac{1}{2N}\sum_{k=0}^{2N-1} Y_{2N}(k) e^{j2\pi kn/2N}$, $n = 0, 1, \cdots, 2N-1$

$$= \frac{1}{2N}\sum_{k=0}^{N/2-1} 2X_N(k) e^{j2\pi kn/2N} + \frac{1}{2N}\sum_{k=3N/2}^{2N-1} 2X_N(k-N) e^{j2\pi kn/2N}$$

$$= \frac{1}{N}\sum_{k=0}^{N/2-1} X_N(k) e^{j2\pi kn/2N} + \frac{1}{N} e^{j\pi n}\sum_{k=N/2}^{N-1} X_N(k) e^{j2\pi kn/2N}$$

If n is even, let $n = 2i$, then

$$y(2i) = \frac{1}{N}\sum_{k=0}^{N/2-1} X_N(k) e^{j2\pi ik/N} + \frac{1}{N}\sum_{k=N/2}^{N-1} X_N(k) e^{j2\pi ik/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_N(k) e^{j2\pi ik/N} = x(i)$$

Hence, $y(2n) = x(n)$, $n = 0, 1, --, N-1$

b. $X_N(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N/2-1}\left(\sum_{i=0}^{N-1} x(i) e^{-j2\pi ki/N}\right) e^{j2\pi kn/2N}$$

$$+ \frac{1}{N} e^{j\pi n}\sum_{k=N/2}^{N-1}\left(\sum_{i=0}^{N-1} x(i) e^{-j2\pi ki/N}\right) e^{j2\pi kn/2N}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\left[\sum_{k=0}^{N/2-1} e^{j\pi k(n-2i)/N} + e^{j\pi n}\sum_{k=N/2}^{N-1} e^{j\pi k(n-2i)/N}\right]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\left[\sum_{k=0}^{N/2-1} e^{j\pi k(n-2i)/N} + e^{j\pi n}\sum_{k=N/2}^{N-1} e^{j\pi k(n-2i)/N}\right]$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\left[1 + e^{j\pi n} e^{j\pi(n-2i)/2}\right]\sum_{k=0}^{N/2-1} e^{j\pi k(n-2i)/N}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\left[1 + e^{j\pi n} e^{j\pi(n-2i)/2}\right]\frac{1 - e^{j\frac{\pi(n-2i)}{N}\cdot\frac{N}{2}}}{1 - e^{j\pi(n-2i)/N}}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\left[1 + e^{-j\pi(n+2i)/2}\right]\frac{1 - e^{j\frac{\pi(n-2i)}{2}}}{1 - e^{j\pi(n-2i)/N}}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\, e^{-j\frac{\pi(n+2i)}{4}} \cos\left[\frac{\pi(n+2i)}{4}\right] \frac{e^{j\frac{\pi(n-2i)}{4}} \sin\frac{\pi(n-2i)}{4}}{e^{j\frac{\pi(n-2i)}{2N}} \sin\frac{\pi(n-2i)}{2N}}$$

$$= \frac{1}{N}\sum_{i=0}^{N-1} x(i)\, e^{-j\pi i} e^{-j\frac{\pi(n-2i)}{2N}} \cos\left[\frac{\pi(n+2i)}{4}\right] \frac{\sin\frac{\pi(n-2i)}{4}}{\sin\frac{\pi(n-2i)}{2N}}$$

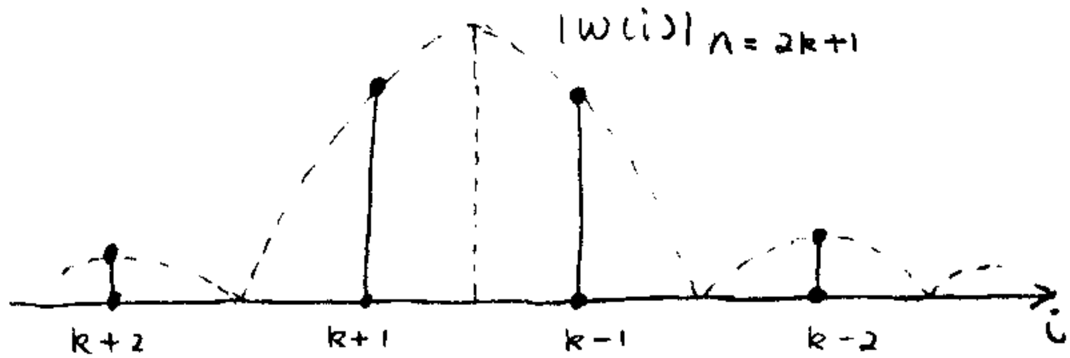
In part a, we have shown that if n is even, i.e., $n=2k$, $k=0,1,\dots,N-1$, then $y(2k)=x(k)$. Now, we let n be odd, i.e., $n=2k+1$, $k=0,1,\dots,N-1$.

Let $$W(i) = \cos\frac{\pi(n+2i)}{4} \; \frac{\sin\frac{\pi(n-2i)}{4}}{\sin\frac{\pi(n-2i)}{2N}}$$

Since. $$\left|\cos\frac{\pi(n+2i)}{4}\right| = \left|\cos\frac{\pi(2k+1+2i)}{4}\right|$$

$$= \left|\cos\left[\frac{(k+i)\pi}{2} + \frac{\pi}{4}\right]\right| = \frac{\sqrt{2}}{2}$$

$$|W(i)| = \frac{\sqrt{2}}{2}\left|\frac{\sin\frac{\pi(n-2i)}{4}}{\sin\frac{\pi(n-2i)}{2N}}\right|$$



The plot of $|W(i)|_{n=2k+1}$ shows that, $y(2k+1)$ depends ~~of~~ on the pair of $x(k+1)$ & $x(k-1)$, $x(k+2)$ & $x(k-2)$, etc. with decreasing weights. Thus, this method acts like the sinc interpolation function.

3. Since $x(n)$ and $y(n)$ are each of length 6, the linear convolution, $x(n)*y(n)$, is of length $6+6-1=11$. Thus, both $z_6(n)$ and $z_9(n)$ are corrupted by aliasing.

Let $z_{11}(n)$, $n=0,1,\cdots,10$, denote the linear convolution, i.e. $z_{11}(n)=x(n)*y(n)$.

From class, we know that.

(1) $z_6(n)=\sum_{n=-\infty}^{\infty} z_{11}(n-6l)$, for $n=0,1,\cdots 5$.

(2) $z_9(n)=\sum_{n=-\infty}^{\infty} z_{11}(n-9l)$, for $n=0,1,\cdots,8$

From (2) we have

$$\left.\begin{array}{l} z_9(0)=z_{11}(0)+z_{11}(9)=9 \\ z_9(1)=z_{11}(1)+z_{11}(10)=12 \end{array}\right\} \text{2 pts. corrupted by aliasing}$$

$$\left.\begin{array}{l} z_9(2)=z_{11}(2)=15 \\ z_9(3)=z_{11}(3)=18 \\ \quad\vdots \\ z_9(8)=z_{11}(8)=6 \end{array}\right\} \text{7 pts. ok}$$

From (1), we have:

$$\left.\begin{array}{l} z_6(0)=z_{11}(0)+z_{11}(6)=21 \\ z_6(1)=z_{11}(1)+z_{11}(7)=21 \\ z_6(2)=z_{11}(2)+z_{11}(8)=21 \\ z_6(3)=z_{11}(3)+z_{11}(9)=21 \\ z_6(4)=z_{11}(4)+z_{11}(10)=21 \end{array}\right\} \text{5 pts. corrupted by aliasing}$$

$z_6(5)=z_{11}(5)=21$, 1 pt ok

Thus, all we need to determine is:

$z_{11}(0)$, $z_{11}(1)$, $z_{11}(9)$, $z_{11}(10)$

Since $z_{11}(6)=15 \Rightarrow z_{11}(0)=z_6(0)-z_{11}(6)=21-15=6$

Since $z_{11}(7)=10 \Rightarrow z_{11}(1)=z_6(1)-z_{11}(7)=21-10=11$

Since $z_{11}(0)=6 \Rightarrow z_{11}(9)=z_9(0)-z_{11}(0)=9-6=3$

Since $z_{11}(1)=11 \Rightarrow z_{11}(10)=z_9(1)-z_{11}(1)=12-11=1$

Aswer:

n	0	1	2	3	4	5	6	7	8	9	10
$z_{11}(n)$	6	11	15	18	20	21	15	10	6	3	1

4. $$X^{12}(k) = \sum_{n=0}^{11} x(n) e^{-j2\pi\frac{kn}{12}}$$
$$= \sum_{m=0}^{3} x(3m) e^{-j2\pi k\frac{3m}{12}} + \sum_{m=0}^{3} x(3m+1) e^{-j2\pi k\frac{(3m+1)}{12}} + \sum_{m=0}^{3} x(3m+2) e^{-j2\pi k\frac{(3m+2)}{12}}$$
$$= \sum_{m=0}^{3} x(3m) e^{-j\frac{2\pi k m}{4}} + e^{-j\frac{2\pi k}{12}}\sum_{m=0}^{3} x(3m+1) e^{-j\frac{2\pi km}{4}} + e^{-j2\pi\frac{2k}{12}}\sum_{m=0}^{3} x(3m+2) e^{-j\frac{2\pi km}{4}}$$

Let $x_0(m) = x(3m)$, $m = 0, 1, 2, 3.$

$x_1(m) = x(3m+1)$, $m = 0, 1, 2, 3.$

$x_2(m) = x(3m+2)$, $m = 0, 1, 2, 3$

$$X^{(12)}(k) = X_0^{(4)}(k) + e^{-j\frac{2\pi k}{12}} X_1^{(4)}(k) + e^{-j\frac{4\pi k}{12}} X_2^{(4)}(k)$$

$$Y^{(4)}(k) = \sum_{n=0}^{3} y(n) e^{-j2\pi\frac{kn}{4}}$$
$$= \sum_{m=0}^{1} y(2m) e^{-j\frac{2\pi k 2m}{4}} + \sum_{m=0}^{1} y(2m+1) e^{-j\frac{2\pi k(2m+1)}{4}}$$

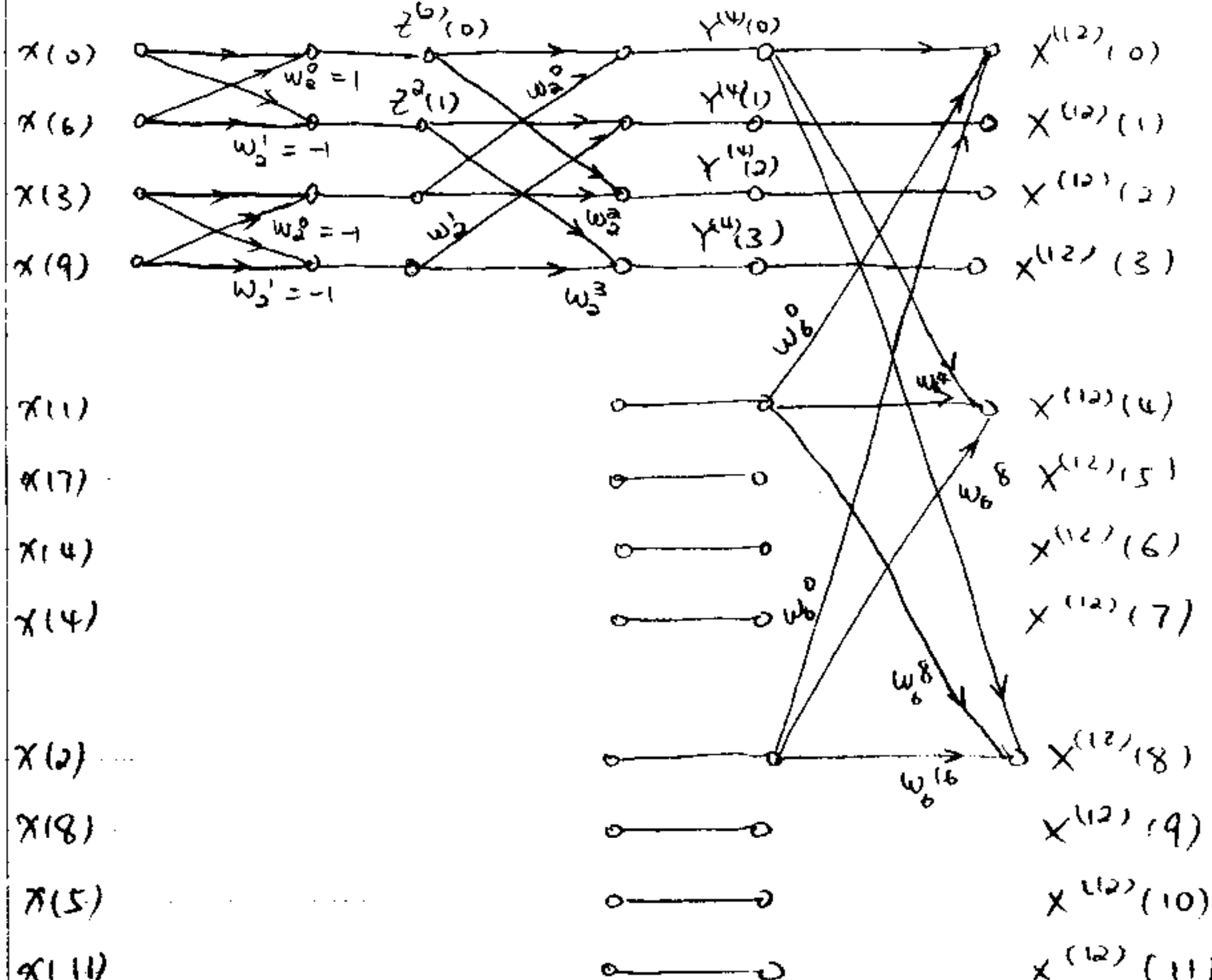
$$= \sum_{m=0}^{1} y(2m) e^{-j\frac{2\pi k m}{2}} + e^{-j\frac{2\pi k}{4}} \sum_{m=0}^{1} y(2m+1) e^{-j\frac{2\pi k m}{2}}$$

Let $y_0(m) = y(2m)$, $m = 0, 1$

$y_1(m) = y(2m+1)$, $m = 0, 1$

$$Y^{(4)}(k) = Y_0^{(2)}(k) + e^{-j2\pi\frac{k}{4}} Y_1^{(2)}(k)$$

$$Z^{(2)}(k) = \sum_{n=0}^{1} z(n) e^{-j2\pi\frac{kn}{2}} = z(0) + e^{-j\pi k} z(1)$$



Where $W_N = e^{-\frac{j2\pi}{N}}$.

5. For simplicity, consider N odd: (results hold for N even as well). Let

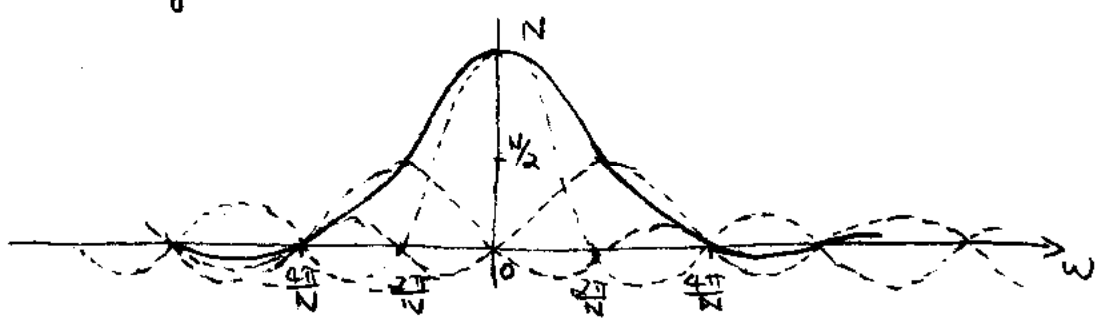
$$w_R(n) = \begin{cases} 1, & |n| < \frac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$, rectangular window of length N

$$\text{DTFT}\{w_R(n)\} = W_R(e^{j\omega}) = \sum_{n=-\frac{N-1}{2}}^{(N-1)/2} e^{-j\omega n} = \frac{\sin(\frac{N}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

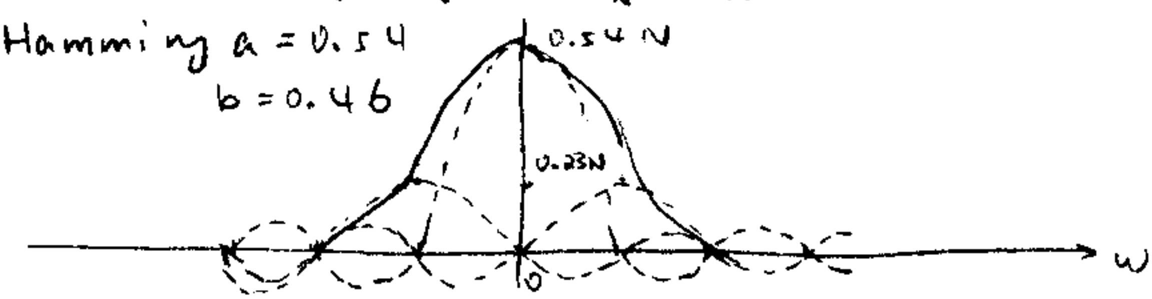
Define $w_c(n) = \cos[\frac{2\pi}{N} n]\, w_R(n)$, then use modulation property and shift property, we have

$$W(e^{j\omega}) = e^{-j\frac{\omega(N-1)}{2}} \left\{ \frac{b}{2} \frac{\sin[\frac{N}{2}(\omega - \frac{2\pi}{N})]}{\sin[\frac{1}{2}(\omega - \frac{2\pi}{N})]} + a\frac{\sin[\frac{N}{2}\omega]}{\sin[\frac{1}{2}\omega]} + \frac{b}{2} \frac{\sin[\frac{N}{2}(\omega + \frac{2\pi}{N})]}{\sin[\frac{1}{2}(\omega + \frac{2\pi}{N})]} \right\}$$

i. Hanning $a=1$, $b=1$



ii. Hamming $a = 0.54$
$b = 0.46$



We notice that the two smaller shifted sinc functions are located at the first zero-crossings of the main sinc.

function and the first (negative) sidelobes of the main sinc are covered by the mainlobes of the shifted sincs. This is why the mainlobe is 50% wider. Also notice that the sidelobes of the shifted sincs and the main sinc always have opposite direction. This is why the sidelobes are lower.

6. Given $X_N(k)$, compute $x(n)$ using a DFT

$$X_N(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{kn}{N}} \quad \text{(the process we have)}$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X_N(k) e^{j2\pi\frac{kn}{N}} \quad \text{(the process we want)}$$

$$= \left[\left[\frac{1}{N}\sum_{k=0}^{N-1} X_N(k) e^{j2\pi\frac{kn}{N}}\right]^*\right]^* \quad (\because x(n) = ((x(n))^*)^*)$$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} \left(X_N(k) e^{j2\pi\frac{kn}{N}}\right)^*\right]^* \quad \left(\begin{array}{c}\text{transfer one } * \\ \text{inside sum}\end{array}\right)$$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} [X_N(k)]^* e^{-j2\pi\frac{kn}{N}}\right]^*$$

If $y(k) = [X_N(k)]^*$, Then

$$Y_N(n) = \sum_{k=0}^{N-1} y(k) e^{-j2\pi\frac{nk}{N}} \quad \text{(the DFT of } y(k)\text{)}$$

Therefore, to get $x(n)$:

① preprocess $X_N(k)$: $y(k) = [X_N(k)]^*$

② take the DFT of $y(k)$

③ post process $Y_N(n)$: $x(n) = \frac{1}{N}[Y_N(n)]^*$

7\. a. $\mathrm{DFT}[V^*(n)] = \sum_{n=0}^{N-1} V^*(n) e^{-j\frac{2\pi n k}{N}}$

$$= \left[\sum_{n=0}^{N-1} V(n) e^{j\frac{2\pi n k}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} V(n) e^{-\frac{j2\pi n N}{N}} e^{j\frac{2\pi n k}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} V(n) e^{-j\frac{2\pi(N-k)n}{N}}\right]^*$$

$$= V_N^*(N-k)$$

b. $V(n) = x(n) + jy(n)$

$V^*(n) = x(n) - jy(n)$

$x(n) = \frac{1}{2}[V(n) + V^*(n)]$

$y(n) = \frac{1}{2j}[V(n) - V^*(n)]$

c. $X_N(k) = \frac{1}{2}[V_N(k) + V^*(N-k)]$

$Y_N(k) = \frac{1}{2j}[V_N(k) - V^*(N-k)]$

where $V(n)$ is constructed as $V(n) = x(n) + jy(n)$ by real signals $x(n)$ and $y(n)$

d. $X^{(N)}(k) = X_0^{(N/2)}(k) + e^{-j\frac{2\pi k}{N}} X_1^{(N/2)}(k)$

where $x_0(n) = x(2n)$, $n = 0, \dots, \frac{N}{2}-1$

$x_1(n) = x(2n+1)$, $n = 0, \dots, \frac{N}{2}-1$

Now, let $V(n) = x_0(n) + jx_1(n)$ be a

N/2 point complex sequence. Then, using the result of part c, $X_0^{(N/2)}(k)$ and $X_1^{(N/2)}(k)$ can be obtained from the N/2 point DFT of $v(n)$.