SPRING 2001

EE438
ASSIGNMENT - 6

→ 1) a) $$E = \int_0^3 \left[x(t) - \hat{x}(t) \right]^2 dt$$

$$= \int_0^3 \left[e^{-1.5t} - a_0 - a_1 t - a_2 t^2 \right]^2 dt$$

• $$\frac{\partial E}{\partial a_1} = -\int_0^3 2 \left[e^{-1.5t} - a_0 - a_1 t - a_2 t^2 \right]^2 dt = 0$$

$$\Rightarrow \int_0^3 \left(a_0 + a_1 t + a_2 t^2 \right) dt = \int_0^3 e^{-1.5t} dt \quad \ldots (i)$$

• $$\frac{\partial E}{\partial a_2} = -\int_0^3 2t \left[e^{-1.5t} - a_0 - a_1 t - a_2 t^2 \right] dt = 0$$

$$\Rightarrow \int_0^3 t \left(a_0 + a_1 t + a_2 t^2 \right) dt = \int_0^3 t e^{-1.5t} dt \quad \ldots (ii)$$

• $$\frac{\partial E}{\partial a_3} = -\int_0^3 2t^2 \left[e^{-1.5t} - a_0 - a_1 t - a_2 t^2 \right] dt = 0$$

$$\Rightarrow \int_0^3 t^2 \left(a_0 + a_1 t + a_2 t^2 \right) dt = \int_0^3 t^2 e^{-1.5t} dt \quad \ldots (iii$$

Now, $$\int_0^3 e^{-1.5t} dt = -\frac{1}{1.5} e^{-1.5t} \Big|_0^3 = -\frac{2}{3} \left[e^{-4.5} - 1 \right]$$

$$\int_0^3 t e^{-1.5t} dt = -\frac{2}{3} t e^{-1.5t} \Big|_0^3 + \frac{2}{3} \int_0^3 e^{-1.5t} dt$$

P.T.O.

$$\therefore \int_0^3 t\,e^{-1.5t}\,dt = -\frac{2}{3}\left[3e^{-4.5} + \frac{2}{3}\left(e^{-4.5} - 1\right)\right]$$

$$= -\frac{22}{9}e^{-4.5} + \frac{4}{9}$$

$$\& \quad \int_0^3 t^2 e^{-1.5t}\,dt = -\frac{2}{3}\left\{ t^2 e^{-1.5t}\Big|_0^3 - \int_0^3 2t\,e^{-1.5t}\,dt \right\}$$

$$= \frac{2}{3}\left[9e^{-4.5} + \frac{44}{9}e^{-4.5} - \frac{8}{9}\right]$$

$$= -\frac{250}{27}e^{-4.5} + \frac{16}{27}$$

Consequently, (i), (ii) and (iii) yield,

$$\left\{\begin{array}{l} 3a_0 + \frac{9}{2}a_1 + 9a_3 = -\frac{2}{3}\left[e^{-4.5} - 1\right] \\ \frac{9}{2}a_0 + 9a_1 + \frac{81}{4}a_2 = -\frac{22}{9}e^{-4.5} + \frac{4}{9} \\ 9a_0 + \frac{81}{4}a_1 + \frac{243}{5}a_2 = -\frac{250}{27}e^{-4.5} + \frac{16}{27} \end{array}\right\}$$

$$\Downarrow$$

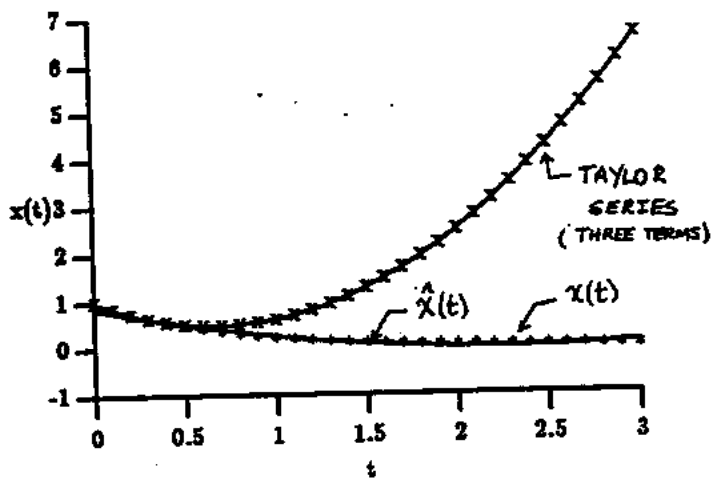
$$\boxed{\begin{array}{l} a_0 = 0.85483655 \\ a_1 = -0.761089 \\ a_2 = 0.16885037 \end{array}}$$

1) • b) The first three terms of the Taylor series are

$$x(0) + x'(0)t + \frac{x''(0)}{2} t^2$$

$$= 1 - 1.5t + 1.125t^2 \;, \text{ using } x(t) = e^{-1.5t} \quad //$$

1) • c)



Recall,

- i) $x(t) := e^{-1.5t}$
- ii) $\hat{x}(t) := a_0 + a_1 t + a_2 t^2$, $a_{0,1,2}$ from 1) a), done earlier.
- iii) Taylor series (three terms) $:= 1 - 1.5t + 1.125t^2$

//

$$1) \cdot d) \quad E = \int_0^3 \left[e^{-1.5t} - \left(a_0 + a_1 t + a_2 t^2\right) \right]^2 dt$$

$$= \int_0^3 \left[e^{-3t} + \left(a_0 + a_1 t + a_2 t^2\right)^2 - 2\left(a_0 + a_1 t + a_2 t^2\right) e^{-1.5t} \right] dt$$

Invoking results from 1) a) and some basic relations

$$= -\frac{1}{3}\left(e^{-9} - 1\right) + \int_0^3 \left[a_0^2 + a_1^2 t^2 + a_2^2 t^4 + 2a_0 a_1 t + 2a_0 a_2 t^2 + 2a_1 a_2 t^3 \right] dt$$

$$- 2a_0\left(-\frac{2}{3}\right)\left[e^{-4.5} - 1\right] - 2a_1\left[-\frac{22}{9} e^{-4.5} + \frac{4}{9}\right]$$

$$-2a_2\left[-\frac{250}{27} e^{-4.5} + \frac{16}{27}\right]$$

$$= -\frac{1}{3}\left(e^{-9} - 1\right) + a_0\left(\frac{4}{3} e^{-4.5} - 1\right) - a_1\left(-\frac{44}{9} e^{-4.5} + \frac{8}{9}\right)$$

$$-a_2\left(-\frac{500}{27} e^{-4.5} + \frac{32}{27}\right) + 3a_0^2 + 9a_1 a_2 + 9\left(a_1^2 + 2a_0 a_2\right)$$

$$+ \frac{81}{2} a_1 a_2 + \frac{243}{5} a_2^2$$

i) $\hat{x}(t)$: Using $a_0 = 0.85483655$, $a_1 = -0.761089$ and,

$a_2 = 0.16885037$ obtained in 1 a) we conclude,

- $E_{\hat{x}(t)} \simeq 0.005118786096914 85$ (to be overly precise)

ii) Taylor series (3 terms retained): Using $a_0 = 1$, $a_1 = -1.5$ and, $a_2 = 1.125$ we arrive at

- $E_{Taylor} \simeq 22.3303670133715$

Clearly, $E_{\hat{x}(t)} << E_{Taylor}$, as intended. //

→ 2) Recall, the definition of $R_n(k)$,

$$R_n(k) \triangleq \sum_{m=-\infty}^{\infty} s_n(m)\, s_n(m-k)$$

& that of E_n,

$$E_n \triangleq \sum_{m=-\infty}^{\infty} \left[s_n(m) - \hat{s}_n(m) \right]^2$$

p^{th} order predictor of $s_n(m)$

$$\triangleq \sum_{m=-\infty}^{\infty} \left[s_n(m) - \sum_{k=1}^{p} \alpha_k s_n(m-k) \right]^2$$

$$= \sum_{m=-\infty}^{\infty} s_n^2(m) - 2 \sum_{m=-\infty}^{\infty} s_n(m) \left\{ \sum_{k=1}^{p} \alpha_k s_n(m-k) \right\}$$

$$+ \sum_{m=-\infty}^{\infty} \left[\left\{ \sum_{k=1}^{p} \alpha_k s_n(m-k) \right\} \left\{ \sum_{l=1}^{p} \alpha_l s_n(m-l) \right\} \right]$$

Contd ... →

Using the definition of $R_n(k)$ after rearranging the terms on the RHS.

$$\doteq R_n(0) - 2\sum_{k=1}^{p}\alpha_k R_n(k) + \sum_{k=1}^{p}\sum_{l=1}^{p}\alpha_k\alpha_l R_n(k-l)$$

(using relation provided in problem statement)

$$\sum_{l=1}^{p}\alpha_l\left\{\sum_{k=1}^{p}\alpha_k R_n(k-l)\right\}$$

$$= R_n(0) - 2\sum_{k=1}^{p}\alpha_k R_n(k) + \sum_{l=1}^{p}\alpha_l R_n(l)$$

Change summation indices (dummy) to match

$$\Rightarrow E_n = R_n(0) - \sum_{k=1}^{p}\alpha_k R_n(k) \text{ , as sought. } //$$

$\rightarrow$ 3) • a) First,

in our case

$$R(0) = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=0}^{\infty}\left(2^{-n} + 4^{-n}\right)^2$$

$$= \sum_{n=0}^{\infty}\left\{2^{-2n} + 2\cdot 8^{-n} + 4^{-2n}\right\}$$

Infinite Geometric series (Summations)

$$\doteq \frac{1}{1-\frac{1}{4}} + \frac{2}{1-\frac{1}{8}} + \frac{1}{1-\frac{1}{16}}$$

$$= \frac{4}{3} + \frac{16}{7} + \frac{16}{15} = \frac{492}{105} = \frac{164}{35}$$

Contd ... $\rightarrow$

Now,

$$R(\ell) \triangleq \sum_{n=-\infty}^{\infty} x(n)x(n-\ell) \underset{\substack{\uparrow \\ \text{Autocorrelation function} \\ \text{is even}}}{=} R(-\ell) \quad , \ \ell = 1, 2, \ldots$$

Consider, then $\ell > 0$,

$$= \sum_{n=-\infty}^{\infty} \left[\left(2^{-n} + 4^{-n}\right)\left(2^{-n+\ell} + 4^{-n+\ell}\right) u(n)u(n-\ell) \right]$$

note lower limit starts at $n=\ell$ ∵ of $u(n)u(n-\ell)$ product above.

$$\overset{\uparrow}{=} \sum_{n=\ell}^{\infty} \left[2^{-2n+\ell} + 4^{-2n+\ell} + 2^{-n} \cdot 4^{-n+\ell} + 4^{-n} \cdot 2^{-n+\ell} \right]$$

$\downarrow$ use $4 = 2^2$

$$= \sum_{n=\ell}^{\infty} \left[2^{-2n+\ell} + 4^{-2n+\ell} + 2^{-3n+2\ell} + 2^{-3n+\ell} \right]$$

let $n = k+\ell$

$$= \sum_{k=0}^{\infty} \left[2^{-2k-\ell} + 4^{-2k-\ell} + 2^{-3k-\ell} + 2^{-3k-2\ell} \right]$$

Geometric series strikes again!

$$= \frac{2^{-\ell}}{1-\frac{1}{4}} + \frac{4^{-\ell}}{1-\frac{1}{16}} + \frac{2^{-\ell}}{1-\frac{1}{8}} + \frac{2^{-2\ell}}{1-\frac{1}{8}}$$

Use $4^{-\ell} = 2^{-2\ell}$ and simplify

$$= \left(\frac{4}{3} + \frac{8}{7}\right)2^{-\ell} + \left(\frac{16}{15} + \frac{8}{7}\right)2^{-2\ell}$$

$$= \left(\frac{52}{21}\right)2^{-\ell} + \left(\frac{232}{105}\right)2^{-2\ell} \quad , \ \ell > 0.$$

$\downarrow$ $\ell = 0$ above

• **NOTE**: $R(0) \overset{\downarrow}{=} \frac{52}{21} + \frac{232}{105} = \frac{492}{105} = \frac{164}{35}$ as attained previously

Finally, $R(\ell) = \left(\frac{52}{21}\right) 2^{-|\ell|} + \left(\frac{232}{105}\right) 4^{-|\ell|} \quad \forall \ell$ //

"for all"

3) • b) $E^{(0)} = R(0) = \frac{164}{35} \simeq 4.6857$

$$K_1 \triangleq \frac{R(1)}{E^{(0)}} = \frac{\left(\frac{52}{21}\right)\cdot 2^{-1} + \left(\frac{232}{105}\right) 4^{-1}}{\left(\frac{164}{35}\right)} \simeq \frac{47}{123}$$

$$\simeq 0.3821 \triangleq \alpha_1^{(1)}$$

Hence, $\hat{x}(n) = 0.3821\, x(n-1)$ //

• c) $E^{(1)} \triangleq \left(1 - K_1^2\right) E^{(0)} \underset{\substack{\uparrow \\ \text{using previous} \\ \text{results}}}{=} \left[1 - \left(\frac{47}{123}\right)^2\right] \frac{164}{35}$

$$\simeq 4.00155$$

$\therefore \quad E^{(1)} \simeq 4.00155$ //

• d) Recall, in the Levinson-Durbin recursion,

$$K_2 \triangleq \frac{R(2) - \alpha_1^{(1)} R(1)}{E^{(1)}}$$

using previous results

$$= \frac{0.7571 - (0.3821)(1.7905)}{4.00155} \simeq 0.01823$$

3) • d) contd...

$$\alpha_1^{(2)} \triangleq \alpha_1^{(1)} - K_2\alpha_1^{(1)} \overset{\text{using past results}}{=} \left(1 - 0.01823\right)0.3821$$

$$\approx 0.375134$$

$$\alpha_2^{(2)} \triangleq K_2 = 0.01823$$

Hence, $\hat{x}(n) = \underbrace{0.375134}_{\alpha_1} x(n-1) + \underbrace{0.01823}_{\alpha_2} x(n-2)$ //

USED LATER in 3) f)

3) • e) $E^{(2)} \triangleq \left(1 - K_2^2\right)E^{(1)}$

$$\overset{\text{using previous results again}}{=} \left(1 - 0.01823^2\right)\left(4.00155\right)$$

$$\Rightarrow E^{(2)} \approx 4.00022$$ //

3) • f) $\hat{x}(n) = \alpha_1 x(n-1) + \alpha_2 (n-2)$

from 3) d) above

$$\hat{x}(n) = 0 \qquad \forall n \leq 0$$

$$\hat{x}(1) = \alpha_1 x(0) = (0.375134)(2) = 0.750268$$

using 3) d) & definition of x(n) (given)

For $n \geq 2$,

$$\hat{x}(n) = \alpha_1 \underbrace{\left[2^{-n+1} + 4^{-n+1}\right]}_{x(n-1)} + \alpha_2 \underbrace{\left[2^{-n+2} + 4^{-n+2}\right]}_{x(n-2)}$$

which simplifies to,

$$\hat{x}(n) = \left(2\alpha_1 + 4\alpha_2\right)2^{-n} + \left(4\alpha_1 + 16\alpha_2\right)4^{-n}$$

$$= \left(0.823188\right)2^{-n} + \left(1.792216\right)4^{-n}$$

using α_1, α_2 from 3) d) above.

Thus,

$$\hat{x}(n) = \begin{cases} 0 & n \leq 0 \\ 0.750268 & n = 1 \\ (0.823188)2^{-n} + (1.792216)4^{-n} & n \geq 2 \end{cases}$$

3) g) $e(n) = x(n) - \hat{x}(n) = 0 \quad \forall\, n < 0$

$$e(0) = x(0) = 2 \qquad (\because \hat{x}(0) = 0)$$

using 3) f) above and x n)

$$e(1) = x(1) - \hat{x}(1) = 0.75 - 0.750268$$

$$= 0.000268$$

And, for $n \geq 2$,

$$e(n) = \left(1 - 0.823188\right)2^{-n} + \left(1 - 1.792216\right)4^{-n}$$

$$e(n) = \left(0.176812\right)2^{-n} - \left(0.792216\right)4^{-n} \qquad n \geq 2.$$

Contd...

Finally,

$$e(n) = \begin{cases} 0 & n < 0 \\ 2 & n = 0 \\ 0.000268 & n = 1 \\ (0.176812)\,2^{-n} - (0.792216)\,4^{-n} & n \geq 2 \end{cases}$$

//

3) h) $E^{(2)} \triangleq \sum_{n=-\infty}^{\infty} \left[x(n) - \hat{x}(n) \right]^2$

$$= 4 + (0.000268)^2 + \sum_{n=2}^{\infty} \left\{ (0.176812)\,2^{-n} - (0.792216)\,4^{-n} \right\}^2$$

$$\simeq 4.000000072 + \sum_{n=2}^{\infty} \left\{ 0.03126248 \cdot 4^{-n} + 0.62760619 \cdot 16^{-n} - 0.28014659 \cdot 8^{-n} \right\}$$

Geometric series results, suitably altered to account for missing $n=0,1$ terms

$$= 4.000000072 + 0.03126248 \left[\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4} \right] + 0.62760619 \left[\frac{1}{1-\frac{1}{16}} - 1 - \frac{1}{16} \right] - 0.28014659 \left[\frac{1}{1-\frac{1}{8}} - 1 - \frac{1}{8} \right]$$

$$\simeq 4.000000072 + 0.002605206 + 0.002615025 - 0.005002617$$

P.T.O.

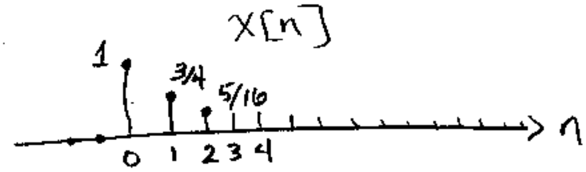
which then yields,

$$E^{(2)} \simeq 4.000217685$$

3) • i) $E^{(2)}$ obtained in h), above, and, e) earlier

are in close agreement.

4) $x[n] = \{2^{-n} + 4^{-n}\} u(n)$



$x(0) = 1$

$x(1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$x(3) = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$

Note:
$u(n-k)u(n-\ell) = 1$ for $n = 2, \ldots, \infty$ if $k \le 2$ and $\ell \le 2$.

a) $\Phi(k,\ell) = \sum_{n=p}^{\infty} x(n-k)\, x(n-\ell)$

For $p = 2$ we have:

$$\Phi(k,\ell) = \sum_{n=2}^{\infty} \{2^{-(n-k)} + \underbrace{4^{-(n-k)}}_{= 2^{-2(n-k)}}\}\{2^{-(n-\ell)} + \underbrace{4^{-(n-\ell)}}_{= 2^{-2(n-\ell)}}\} u(n-k)u(n-\ell)$$

For $k, \ell \le 2$ we have:

$$\Phi(k,\ell) = \sum_{n=2}^{\infty} \{2^{-2n+k+\ell} + 2^{-3n+2k+\ell} + 2^{-3n+k+2\ell} + 2^{-4n+2k+2\ell}\}$$

An aside

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for } |a| < 1$$

$$\sum_{n=0}^{\infty} a^n - a^0 - a^1 = \frac{1}{1-a} - a^0 - a^1$$

$$\sum_{n=2}^{\infty} a^n = \frac{1}{1-a} - (1+a) = \frac{1 - (1+a)(1-a)}{1-a}$$

$$= \frac{1 - (1 - a^2)}{1-a} = \frac{a^2}{1-a}$$

$$\Phi(k,l) = \sum_{n=2}^{\infty} \underbrace{(2^{-2})^n}_{a=\frac{1}{4}} 2^{k+l} + \sum_{n=2}^{\infty} \underbrace{(2^{-3})^n}_{a=\frac{1}{8}} 2^{2k+l} + \sum_{n=2}^{\infty} (2^{-3})^n 2^{k+2l}$$

$$+ \sum_{n=2}^{\infty} \underbrace{(2^{-4})^n}_{a=\frac{1}{16}} 2^{2k+2l}$$

$$= \frac{\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}} 2^{k+l} + \frac{\left(\frac{1}{8}\right)^2}{1-\frac{1}{8}}\left(2^{2k+l} + 2^{k+2l}\right) + \frac{\left(\frac{1}{16}\right)^2}{1-\frac{1}{16}} 2^{2k+2l}$$

$$= \frac{\left(\frac{1}{4}\right)^2}{3/4} 2^{k+l} + \frac{\left(\frac{1}{8}\right)^2}{7/8}\left(2^{2k+l} + 2^{k+2l}\right) + \frac{\left(\frac{1}{16}\right)^2}{15/16} 2^{2k+2l}$$

$$= \left(\frac{1}{4\cdot 3}\right) 2^{k+l} + \left(\frac{1}{8\cdot 7}\right)\left(2^{2k+l} + 2^{k+2l}\right) + \left(\frac{1}{16\cdot 15}\right) 2^{2k+2l} \quad \text{for } k,l \leq 2$$

b) $$E^{(2)} = \sum_{n=2}^{\infty} \left\{ x(n) - \hat{x}(n) \right\}^2$$

$$= \sum_{n=2}^{\infty} \left\{ x(n) - \alpha_1^{(2)} x(n-1) - \alpha_2^{(2)} x(n-2) \right\}^2$$

i) $$\frac{\partial}{\partial \alpha_1^{(2)}} E^{(2)} = 0 = 2 \sum_{n=2}^{\infty} \left\{ x(n) - \alpha_1^{(2)} x(n-1) - \alpha_2^{(2)} x(n-2) \right\} (-x(n-1))$$

ii) $$\frac{\partial}{\partial \alpha_2^{(2)}} E^{(2)} = 0 = 2 \sum_{n=2}^{\infty} \left\{ x(n) - \alpha_1^{(2)} x(n-1) - \alpha_2^{(2)} x(n-2) \right\} (-x(n-2))$$

from i) $\sum_{n=2}^{\infty} x(n)\,x(n-1) - \alpha_1^{(2)} \sum_{n=2}^{\infty} x(n-1)\,x(n-1) - \alpha_2^{(2)} \sum_{n=2}^{\infty} x(n-2)\,x(n-1) = 0$

$$\Phi(0,1) - \alpha_1^{(2)}\,\Phi(1,1) - \alpha_2^{(2)}\,\Phi(2,1) = 0 \qquad (1)$$

from ii) $\Phi(0,2) - \alpha_1^{(2)}\,\Phi(1,2) - \alpha_2^{(2)}\,\Phi(2,2) = 0 \qquad (2)$

$$\Phi(0,1) = \left(\frac{1}{4\cdot 3}\right)2 + \left(\frac{1}{7\cdot 8}\right)(2+2^2) + \left(\frac{1}{15\cdot 16}\right)2^2$$

$$= \frac{1}{2\cdot 3} + \frac{\cancel{6}\,3}{7\cdot\cancel{8}\,4} + \frac{\cancel{4}}{15\cdot\cancel{16}\,4}$$

$$= \frac{1}{2\cdot 3} + \frac{3}{7\cdot 2\cdot 2} + \frac{1}{3\cdot 5\cdot 2\cdot 2} = \frac{2\cdot 5\cdot 7 + 3\cdot 3\cdot 5 + 7}{2\cdot 2\cdot 3\cdot 5\cdot 7}$$

$$= \frac{70+45+7}{60\cdot 7} = \frac{122}{420} = \frac{61}{210}$$

$$\Phi(1,1) = \frac{\cancel{4}}{\cancel{4}\cdot 3} + \frac{1}{\cancel{8}\cdot 7}\overset{2,\cancel{8}}{(2^3+2^3)} + \left(\frac{1}{\cancel{16}\cdot 15}\right)\cancel{2^4}$$

$$= \frac{1}{3} + \frac{2}{7} + \frac{1}{3\cdot 5} = \frac{5\cdot 7 + 3\cdot 5\cdot 2 + 7}{3\cdot 5\cdot 7}$$

$$= \frac{35+30+7}{3\cdot 5\cdot 7} = \frac{72}{105} = \frac{144}{210}$$

$$\Phi(2,1) = \frac{188}{105} = \frac{376}{210} = \Phi(1,2)$$

$$\Phi(0,2) = \frac{1272}{1680} = \frac{159}{210}$$

$$\Phi(2,2) = \frac{492}{105} = \frac{984}{210}$$

$$\begin{bmatrix} \Phi(1,1) & \Phi(2,1) \\ \Phi(1,2) & \Phi(2,2) \end{bmatrix} \begin{bmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{bmatrix} = \begin{bmatrix} \Phi(0,1) \\ \Phi(0,2) \end{bmatrix}$$

Using Cramer's rule:

$$\alpha_1^{(2)} = \frac{\begin{vmatrix} \Phi(0,1) & \Phi(2,1) \\ \Phi(0,2) & \Phi(2,2) \end{vmatrix}}{\Delta} = \Phi(0,1)\Phi(2,2) - \Phi(0,2)\Phi(2,1)$$

$$\alpha_2^{(2)} = \frac{\begin{vmatrix} \Phi(1,1) & \Phi(0,1) \\ \Phi(1,2) & \Phi(0,2) \end{vmatrix}}{\Delta} = \Phi(1,1)\Phi(0,2) - \Phi(1,2)\Phi(0,1)$$

$$\Delta = \Phi(1,1)\Phi(2,2) - \Phi(1,2)\Phi(2,1)$$

$$= \Phi(1,1)\Phi(2,2) - \Phi^2(1,2)$$

$$\alpha_1^{(2)} = \frac{\left(\frac{61}{\cancel{210}}\right)\frac{984}{\cancel{210}} - \frac{159}{\cancel{210}}\cdot\frac{376}{\cancel{210}}}{\frac{144}{\cancel{210}}\,\frac{984}{\cancel{210}} - \frac{376}{\cancel{210}}\cdot\frac{376}{\cancel{210}}} = \frac{240}{320} = \frac{3}{4}$$

$$\alpha_2^{(2)} = \frac{\frac{144}{\cancel{210}}\,\frac{159}{\cancel{210}} - \frac{376}{\cancel{210}}\cdot\frac{61}{\cancel{210}}}{\frac{144}{\cancel{210}}\cdot\frac{984}{\cancel{210}} - \frac{376}{\cancel{210}}\cdot\frac{376}{\cancel{210}}} = \frac{-40}{320} = -\frac{1}{8}$$

So $\quad \hat{x}(n) = \frac{3}{4}x(n) - \frac{1}{8}x(n-1)$

c) $$E^{(2)} = \sum_{n=2}^{\infty}\left(x(n) - \alpha_1^{(2)}x(n-1) - \alpha_2^{(2)}x(n-2)\right)\left(x(n) - \alpha_1^{(2)}x(n-1) - \alpha_2^{(2)}x(n-2)\right)$$

$$= \sum_{n=2}^{\infty} x(n)\left(x(n) - \alpha_1^{(2)}x(n-1) - \alpha_2^{(2)}x(n-2)\right)$$

$$- \alpha_1^{(2)} \underbrace{\sum_{n=2}^{p} x(n-1)\left(x(n) - \alpha_1^{(2)}x(n-1) - \alpha_2^{(2)}x(n-2)\right)}_{=0 \text{ (see eqn (i))}}$$

$$- \alpha_2^{(2)} \underbrace{\sum_{n=2}^{p} x(n-2)\left(x(n) - \alpha_1^{(2)}x(n-1) - \alpha_2^{(2)}x(n-2)\right)}_{=0 \text{ see eqn (ii)}}$$

$$= \Phi(0,0) - \alpha_1^{(2)}\Phi(0,1) - \alpha_2^{(2)}\Phi(0,2)$$

$$= \left(\frac{1}{\underset{2\cdot2\cdot3}{4\cdot3}} + \frac{2}{\underset{7\cdot2\cdot2\cdot2}{7\cdot8}} + \frac{1}{\underset{3\cdot5\cdot2\cdot2\cdot2\cdot2}{15\cdot16}}\right) - \frac{3}{4}\cdot\frac{61}{210} + \frac{1}{8}\,\frac{159}{210}$$

$$= \frac{4\cdot5\cdot7+2\cdot3\cdot5\cdot2+7}{2\cdot2\cdot2\cdot2\cdot3\cdot5\cdot7} - \frac{3\cdot61\cdot2}{8\cdot210} + \frac{159}{8\cdot210}$$

$$= \frac{4\cdot5\cdot7+3\cdot5\cdot4+7 - 6\cdot61 + 159}{8\cdot210} =$$

$$= \frac{140+60+7-366+159}{8\cdot210} = \frac{366-366}{8\cdot210} = 0$$

d) We just found that a 2nd order predictor predicts our sequence perfectly, giving zero error. Adding a third term to our predictor will not improve the prediction any (and it won't hurt it either). We'll still get zero error. Solving for the 3rd order predictor as we did in parts a) and b) gives us a system of 3 equations and 3 unknowns. Yet the equations are not linearly independent. Thus there are an infinite number of solutions. One of the solutions, certainly, is

$$a_1^{(3)} = a_1^{(2)} = \frac{3}{4}$$
$$a_2^{(3)} = a_2^{(2)} = -\frac{1}{8}$$
$$a_3^{(3)} = 0$$

As we found in part c), this solution gives $E^{(3)} = 0$.

e) The autocorrelation method tries to predict values of $\hat{x}[n]$ for $n < p$ where it doesn't have a full set of p past non-zero values. Thus, there will always be some error associated with it. The covariance method, however, does not try to predict these values. So its error will be smaller. In our case, the covariance method gave zero error for $p > 1$.