Last compiled: December 5, 2000
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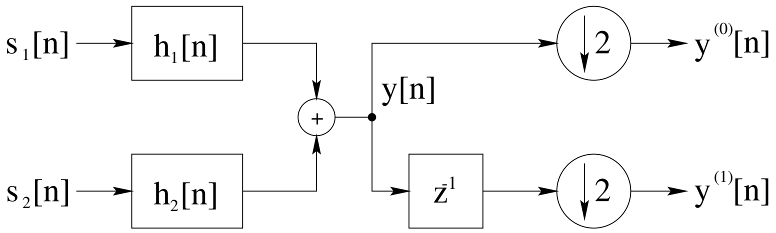

Figure 1: Two signals are sent over two channel and received as a sum. Then processed.

1 In General

Assume that a system has transmit antennas and one receive antenna. The first antenna transmits the sequence

$$s_1[n] = \{s[n], -s[n+1], s[n+2], -s[n+3], \dots\}, \tag{1}$$

while the second antenna transmits the sequence

$$s_2[n] = \{s[n+1], s[n], s[n+3], s[n+2], \dots\}. \tag{2}$$

The two signals $s_1[n]$ and $s_2[n]$ pass through (possibly time varying) channels $h_1[n]$ and $h_2[n]$, respectively, in arriving at the single receive antenna. The received signal is

$$y[n] = s_1[n] * h_1[n] + s_2[n] * h_2[n], \tag{3}$$

where * denotes convolution.

Two output sequences are formed from the received signal. The first is formed by downsampling $y[n]$ by a factor of two. The second is formed by delaying $y[n]$ by one and then downsampling by two. Thus, as shown graphically in Figure 1, the ouput sequences are

$$y^{(0)}[n] = y[2n] \tag{4}$$

and

$$y^{(1)}[n] = y[2n+1]. \tag{5}$$

As will be shown shortly, the output sequences $y^{(0)}[n]$ and $y^{(1)}[n]$ may be expressed in terms of two subsequences formed from $s[n]$, namely

$$s^{(0)}[n] = s[2n] \tag{6}$$

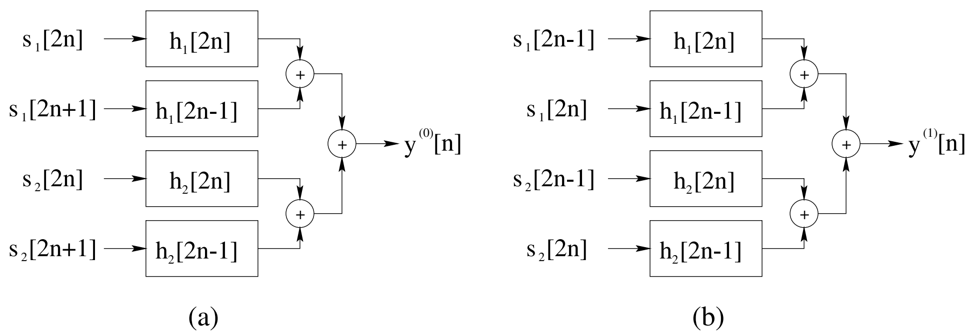

Figure 2: Polyphase implementation of $y^{(0)}[n]$ and $y^{(1)}[n]$.

and

$$s^{(1)}[n] = s[2n+1]. \tag{7}$$

The filtering and downsampling operations used to form $y^{(0)}[n]$ and $y^{(1)}[n]$ can be done efficiently using a polyphase implementation (c.f. [PM96]). The polyphase implementations are

$$\begin{aligned} y^{(0)}[n] &= s_1[2n] * h_1[2n] + s_1[2n+1] * h_1[2n-1] \\ &\quad + s_2[2n] * h_2[2n] + s_2[2n+1] * h_2[2n-1] \end{aligned} \tag{8}$$

$$\begin{aligned} y^{(1)}[n] &= y[2n-1] \\ &= s_1[2n-1] * h_1[2n] + s_1[2n] * h_1[2n-1] \\ &\quad + s_2[2n-1] * h_2[2n] + s_2[2n] * h_2[2n-1]. \end{aligned} \tag{9}$$

See Figure 1 for a graphical description of the polyphase representation of $y^{(0)}[n]$. From Equations 1 and 2 we see that

$$s_1[2n] = s[2n] = s^{(0)}[n], \tag{10}$$

$$s_1[2n+1] = -s[2n+1] = -s^{(1)}[n], \tag{11}$$

$$s_2[2n] = s[2n+1] = s^{(1)}[n], \tag{12}$$

and

$$s_2[2n+1] = s[2n] = s^{(0)}[n]. \tag{13}$$

Furthermore, we see that

$$\begin{aligned} s_i[2n-1] * h_i[2n] =& s_i[2n+1] * h_i[2n-2] \\ =& s_i[2n+1] * h_i[2(n-1)] \\ =& s_i[2n+1] * h_i^{(0)}[n-1]. \end{aligned} \tag{14}$$

Thus, denoting $h_i[2n-m]$ as $h_i^{(m)}[n]$, we have

$$\begin{aligned} y^{(0)}[n] =& s^{(0)}[n] * (h_1^{(0)}[n] + h_2^{(1)}[n]) \\ &+ s^{(1)}[n] * (-h_1^{(1)}[n] + h_2^{(0)}[n]) \\ =& s^{(0)}[n] * f_1[n] + s^{(1)}[n] * f_2[n], \end{aligned} \tag{15}$$

where $f_1[n] = h_1^{(0)}[n] + h_2^{(1)}[n]$ and $f_2[n] = -h_1^{(1)}[n] + h_2^{(0)}[n]$, and

$$\begin{aligned} y^{(1)}[n] =& s_1[2n+1] * h_1^{(0)}[n-1] + s_1[2n] * h_1^{(1)}[n] \\ &+ s_2[2n+1] * h_2^{(0)}[n-1] + s_2[2n] * h_2^{(1)}[n] \\ =& s^{(0)}[n] * (h_1^{(1)}[n] + h_2^{(0)}[n-1]) \\ &+ s^{(1)}[n] * (-h_1^{(0)}[n-1] + h_2^{(1)}[n]) \\ =& s^{(0)}[n] * f_3[n] + s^{(1)}[n] * f_4[n], \end{aligned} \tag{16}$$

where $f_3[n] = h_1^{(1)}[n] + h_2^{(0)}[n-1]$ and $f_4[n] = -h_1^{(0)}[n-1] + h_2^{(1)}[n]$.

2 Summary

$$y^{(0)}[n] = s^{(0)}[n] * f_1[n] + s^{(1)}[n] * f_2[n] \tag{17}$$
$$y^{(1)}[n] = s^{(0)}[n] * f_3[n] + s^{(1)}[n] * f_4[n], \tag{18}$$

where

$$\begin{array}{lcccc} f_1[n] & = & h_1^{(0)}[n] & + & h_2^{(0)}[n] \\ f_2[n] & = & -h_1^{(1)}[n] & + & h_2^{(0)}[n] \\ f_3[n] & = & h_1^{(1)}[n] & + & h_2^{(0)}[n-1] \\ f_4[n] & = & -h_1^{(0)}[n-1] & + & h_2^{(1)}[n]. \end{array} \tag{19}$$

3 Matrix Notation

Matrix notation provides a convenient way of representing the system outputs that will be needed for equalization. The length N_g vectors of outputs

$$\mathbf{y}^{(i)} = [\ y^{(i)}[n] \quad \ldots \quad y^{(i)}[n-Ng+1]\ ]^T \qquad i = 1,2 \tag{20}$$

may be formed using convolution matrices. If the impulse response $f_j[n]$ is non–zero for $n = 0, \ldots, L-1$ only, then its $N_g \times (L + N_g - 1)$ convolution matrix is

$$\mathbf{F_j} = \begin{bmatrix} f_j[0] & \ldots & f_j[L-1] & 0 & \ldots & & & 0 \\ 0 & f_j[0] & \ldots & f_j[L-1] & 0 & \ldots & & 0 \\ 0 & 0 & f_j[0] & \ldots & f_j[L-1] & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ldots & & & 0 & f_j[0] & \ldots & f_j[L-1] \end{bmatrix}.$$

Letting

$$\mathbf{F} = \begin{bmatrix} \mathbf{F_1} & \mathbf{F_2} \\ \mathbf{F_3} & \mathbf{F_4} \end{bmatrix}, \tag{21}$$

we may write

$$\mathbf{y} = \mathbf{F}\mathbf{s}, \tag{22}$$

where

$$\mathbf{y} = [\ \mathbf{y}^{(0)^T} \quad \mathbf{y}^{(1)^T}\ ]^T, \tag{23}$$

$$\mathbf{s}^{(i)} = [\ s^{(i)}[n] \quad \ldots \quad s^{(i)}[n-L-Ng]\ ]^T \qquad i = 0,1 \tag{24}$$

and

$$\mathbf{s} = [\ \mathbf{s}^{(0)^T} \quad \mathbf{s}^{(1)^T}\ ]^T. \tag{25}$$

4 Symbol Recovery Through Equalization

Symbol estimates $\hat{s}^{(0)}[n-D]$ and $\hat{s}^{(1)}[n-D]$ are formed by choosing length N_g equalizing vectors $\mathbf{g_i} = \left[\ g_i[0] \quad g_i[1] \quad \ldots \quad g_i[N_g-1]\ \right]^T$, $i = 1, \ldots, 4$, according to some criterion. The recoverd signals are

$$\begin{aligned} \hat{s}^{(0)}[n-D] &= g_1[n] * y^{(0)}[n] + g_2[n] * y^{(1)}[n] \\ \hat{s}^{(1)}[n-D] &= g_3[n] * y^{(0)}[n] + g_4[n] * y^{(1)}[n] \end{aligned}$$

The elements of $\mathbf{g_i}$, $i = 1, \ldots, 4$, are chosen to minimize the means squared error between the sent signals and the estimated signals. Denoting the vector of equalizer taps as $\mathbf{g}_i = [\ g_i[0], \ldots, g_i[Ng-1]\ ]^T$, $i = 1, \ldots, 4$, the estimates of the signals are

$$\hat{s}^{(0)}[n-D] = \mathbf{g_{12}}^H \mathbf{y} \tag{26}$$

and

$$\hat{s}^{(1)}[n-D] = \mathbf{g_{34}}^H \mathbf{y}, \tag{27}$$

where $\mathbf{g_{12}} = [\ \mathbf{g_1}^T \quad \mathbf{g_2}^T\ ]^T$ and $\mathbf{g_{34}} = [\ \mathbf{g_3}^T \quad \mathbf{g_4}^T\ ]^T$.

The minimum mean-square error error (MMSE) equalizers are the solutions to

$$\mathbf{g_{12}} = \arg\min_{\mathbf{g}} \quad E\left\{\left|\mathbf{g}\mathbf{y}[n] - s^{(0)}[n-D]\right|^2\right\} \tag{28}$$

and

$$\mathbf{g_{34}} = \arg\min_{\mathbf{g}} \quad E\left\{\left|\mathbf{g}\mathbf{y}[n] - s^{(1)}[n-D]\right|^2\right\}. \tag{29}$$

where $\mathbf{F}_{ij} = [\mathbf{F}_i \quad \mathbf{F}_j]$, where Rnn is the $2N_g \times 2N_g$ noise convariance matrix, and where δ_D is a $Ng + L - 1$–lengthed column vector of zeros with a one in the Dth position.

Expressed in terms of the channel impulse responses $h_1[n]$ and $h_2[n]$, the solutions to (28) and (29) are

$$\mathbf{g_{12}} = (\mathbf{F}\mathbf{F}^H + \mathbf{R}_{\eta\eta})^{-1}\mathbf{F_{13}}\delta_{\mathbf{D}} \tag{30}$$

and

$$\mathbf{g_{34}} = (\mathbf{F}\mathbf{F}^H + \mathbf{R}_{\eta\eta})^{-1}\mathbf{F_{24}}\delta_{\mathbf{D}}. \tag{31}$$

These solutions assume that the symbols are white and i.i.d. with unit variance.

5 Observations

For best results, We must have $D > 0$. Otherwise $g12$ and $g34$ are linearly related, and the BER is extremely high. If $D = 0$, there is no contribution to the equalizers from $h_2[n]$.. It makes sense that D must be one or greater because $y[n]$ is delayed by one sample when forming $y^{(1)}[n]$, so we need to delay by one sample before forming an estimate. Is there a way around this? What about a different scheme for $s_1[n]$ and $s_2[n]$?

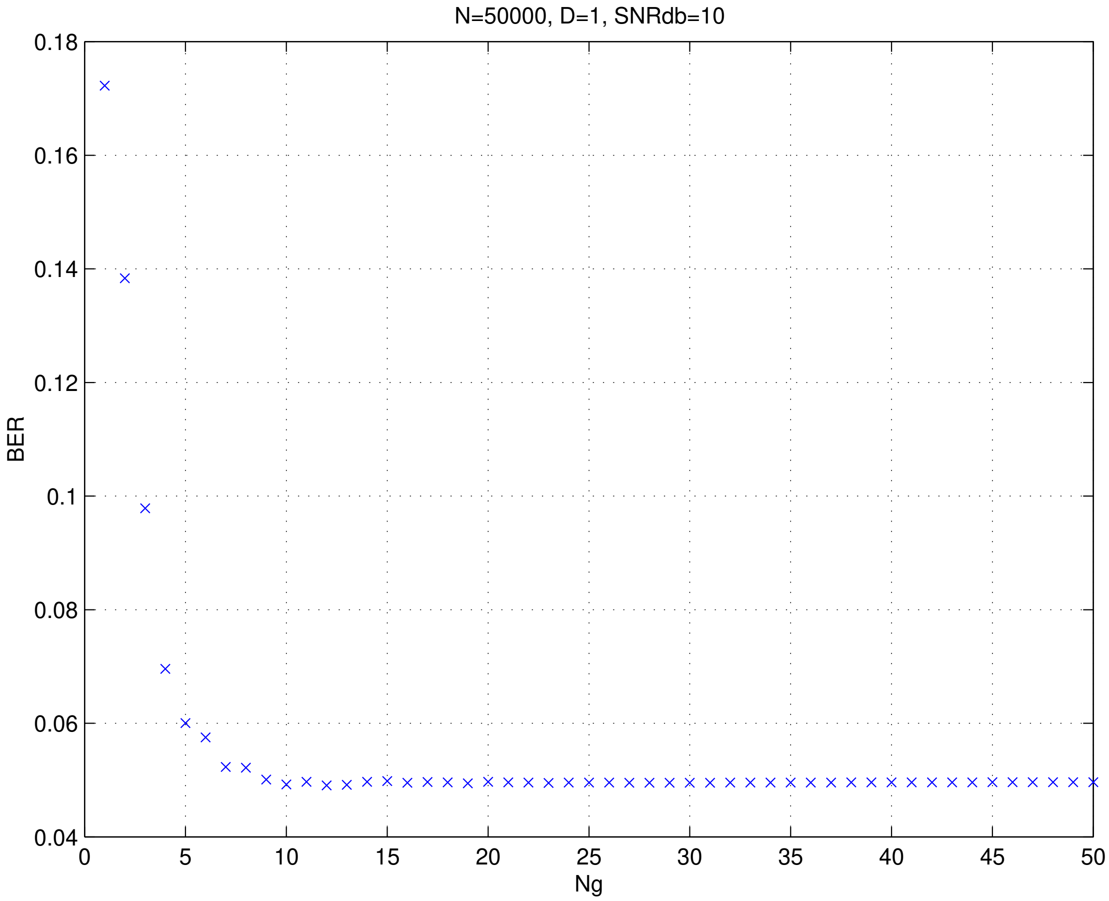

Figure 3: Title on figure

Figure 3 shows BER vs. equalizer length N_g for channels of length 4. (In this case, $h_1 = [1 \quad 1 \quad 1 \quad 1]$ and $h_2 = [1 \quad 1 \quad -1 \quad -1]$.) We do indeed see that the BER curve flattens out when N_g is approximately three times the length of the channel's impulse response (i.e., at about $N_g = 12$). Thus, similar to previous work by Tom and Prof. Zoltowski, it appears that equalizers of length $3 * L$ are adequate, even though an IIR filter is required in theory.

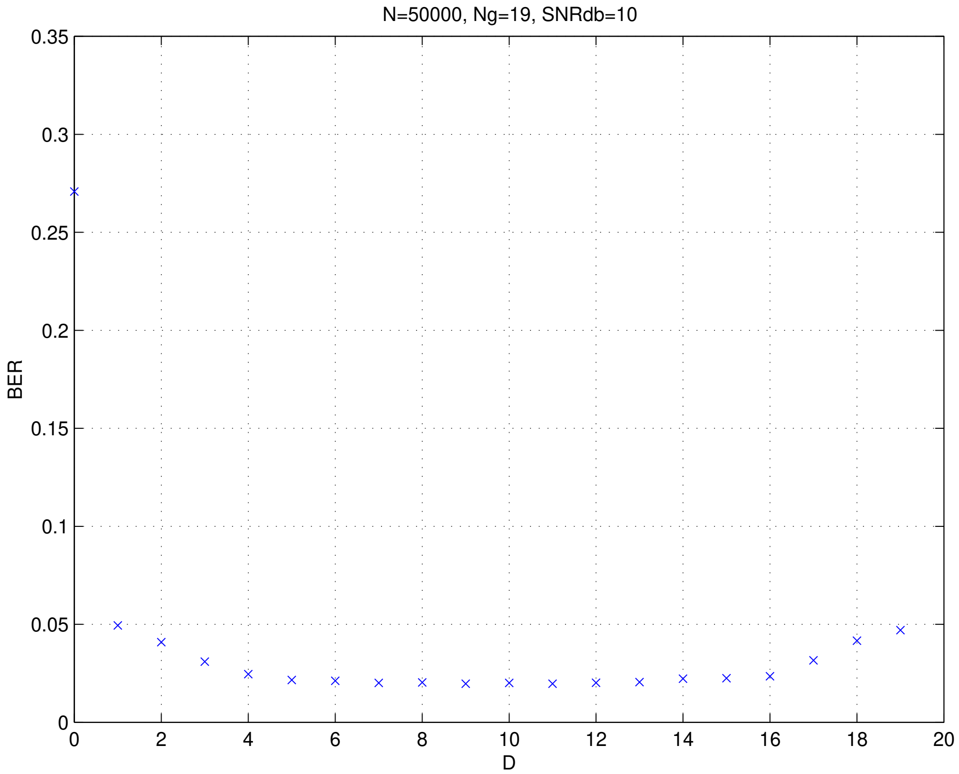

Figure 4: Title on figure

Figure 4 shows BER vs. equalizer delay D. Unlike Tom's work where he noticed little difference in BER for various delays D (see his most recent journal article–the one that has Bill listed as an author [KHZ]), Figure 4 shows that BER is best for $L < D < N_g - L$. However, within this range there is little difference in BER. Thus, when trading delay for BER it is best to choose $D = L + 1$ if the system can tolerate such a delay.

The equalizers can be written entirely in term of $h_1[n]$ and $h_2[n]$ and the noise variance. We had hoped to find some symmetry buried in the structure of the equalizers, but it does not appear that there is any. Analytic analysis for simple cases (such as for $Ng = 1$) give little hope for finding the desired symmetry. For the channels described above and for $D = 1$, the length 12 equalizers are

n	g_1	g_2	g_3	g_4
0	0.0150	0.3792	−0.1752	−0.0588
1	−0.0652	−0.1568	0.3729	−0.5912
2	0.3848	−0.5362	0.3548	−0.1581
3	0.3086	−0.1190	−0.0361	0.1899
4	−0.0262	0.1431	−0.0570	−0.0396
5	−0.0197	−0.0683	0.1346	−0.1995
6	0.1333	−0.1747	0.1086	−0.0353
7	0.0906	−0.0232	−0.0213	0.0649
8	−0.0162	0.0463	−0.0155	−0.0188
9	−0.0038	−0.0239	0.0437	−0.0587
10	0.0369	−0.0421	0.0297	−0.0097
11	0.0186	−0.0046	−0.0021	0.0089

Even with great symmetry in $h_1[n]$ and $h_2[n]$, as we have above, the equalizers are completely different.

References

[KHZ] Thomas P. Krauss, William J. Hillery, and Michael D. Zoltowski. "Downlink Specific Linear Equalization for Frequency Selective CDMA Cellular Systems". *IEEE ???*, ???(???):???, ??? ???

[PM96] John G. Proakis and Dimitris G. Manolakis. *Digital Signal Processing: Principles, Algorithms, and Applications.* Prentice Hall, third edition, 1996.

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