ECE600 Random Variables and Waveforms
Spring 2024

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MSEE 336

Homework Assignment #10
Should be completed by the Final Exam

Reading Assignment: Read Sections 7-4, 9-1 and 9-2 of Papoulis.

1. (*Papoulis* 7-20) We place at random n points in the interval $(0,1)$ and we denote by $\mathbf{X}$ and $\mathbf{Y}$ the distance from 0 to the first (closest) and last (furthest) point, respectively. Find $F_{\mathbf{X}}(x)$, $F_{\mathbf{Y}}(y)$, and $F_{\mathbf{XY}}(x,y)$.

2. (*Papoulis* 7-25) Show that if $a_n \to a$ and $E[|\mathbf{X}_n - a_n|^2] \to 0$, then $\mathbf{X}_n \to a$ in mean-square as $n \to \infty$.

3. (*Papoulis* 7-27) An infinite sum is by definition a limit:

$$\sum_{k=1}^{\infty} \mathbf{X}_k = \lim_{n\to\infty} \mathbf{Y}_n, \qquad \text{where} \quad \mathbf{Y}_n = \sum_{k=1}^{n} \mathbf{X}_k.$$

   Show that if the random variables $\mathbf{X}_k$ are independent with zero-mean and variances σ_k^2, then the infinite sum exists in the mean-square sense iff

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty.$$

   Hint:

$$E[(\mathbf{Y}_{n+m} - \mathbf{Y}_n)^2] = \sum_{k=n+1}^{n+m} \sigma_k^2.$$

4. Consider a random experiment $\{\mathcal{S}, \mathcal{F}, P\}$ with sample space $\mathcal{S} = \{1, 2, 3, \ldots\}$ and associated pmf $p(n) = P(\{\omega = n\}) = \alpha/n^2$. Consider the sequence $\{\mathbf{X}_n\}$ of random variables defined by

$$\mathbf{X}_n(\omega) = \begin{cases} n, & \text{if } \omega = n, \\ 0, & \text{if } \omega \neq n. \end{cases}$$

   Prove that $\{\mathbf{X}_n\}$ converges *almost everywhere* to $\mathbf{X} = 0$, but does not converge in the mean square sense; that is $\mathrm{E}\{|\mathbf{X}_n - 0|^2\}$ does not converge to 0 as $n \to \infty$.

5. (*Papoulis* 7-13) The random variables $\{\mathbf{X}_k\}$ are i.i.d. with moment generating function $\phi_{\mathbf{X}}(s) = E[e^{i\omega \mathbf{X}}]$. The random variable $\mathbf{N}$ takes on values $0, 1, 2 \ldots$, and it's discrete moment function $\Gamma_{\mathbf{N}}(z)$ is defined as

$$\Gamma_{\mathbf{N}}(z) = E[z^{\mathbf{X}}] = \sum_{n=0}^{\infty} p_{\mathbf{N}}(n) z^n,$$

where $p_{\mathbf{N}}(n)$ is the p.m.f. of $\mathbf{N}$.

(a) Show that if

$$\mathbf{Y} = \sum_{j=1}^{\mathbf{N}} \mathbf{X}_j,$$

then

$$\phi_{\mathbf{Y}}(s) = E[e^{s\mathbf{Y}}] = \Gamma_{\mathbf{N}}(\phi_{\mathbf{X}}(s)).$$

Hint: $E[e^{s\mathbf{Y}} | \mathbf{N} = k] = E[e^{s(\mathbf{X}_1 + \ldots + \mathbf{X}_k)}] = \phi^k_{\mathbf{X}}(s)$.

(b) *Special Case:* Show that if $\mathbf{N}$ is Poisson with mean a, then

$$\phi_{\mathbf{Y}}(s) = e^{a(\phi_{\mathbf{X}}(s) - 1)}.$$

6. (*Papoulis* 9-1) Assume we toss a fair coin. If "heads" occurs, $\mathbf{X}(t) = \sin 2\pi t$; if "tails" occurs, set $\mathbf{X}(t) = 2t$.
   (a) Find $E[\mathbf{X}(t)]$.
   (b) Find $F_{\mathbf{X}(t)}(x)$ for $t = 0.25$, $t = 0.5$, and $t = 1$.

7. (*Papoulis* 9-2) The random process

$$\mathbf{X}(t) = e^{\mathbf{A}t}$$

is a family of exponentials depending on the random variable $\mathbf{A}$. Express the mean function $\eta_{\mathbf{X}}(t)$, the autocorrelation function $R_{\mathbf{XX}}(t_1, t_2)$, and the first order density function $f_{\mathbf{X}(t)}(x)$ of $\mathbf{X}(t)$ in terms of the density function $f_{\mathbf{A}}(a)$ of $\mathbf{A}$.

8. (*Papoulis* 9-3) Suppose that $\mathbf{X}(t)$ is a homogeneous Poisson counting process (this will be discussed in the last lecture) such that $E[\mathbf{X}(9)] = 6$.
   (a) Find the mean and variance of $\mathbf{X}(8)$.
   (b) Find $P(\{\mathbf{X}(2) \leq 3\})$.
   (c) Find $P(\{\mathbf{X}(4) \leq 5\} | \{\mathbf{X}(2) \leq 3\})$.

9. (*Papoulis* 9-8) The random process $\mathbf{X}(t)$ is WSS and Gaussian with mean $E[\mathbf{X}(t)] = 0$ and autocorrelation $R_{\mathbf{X}}(\tau) = 4e^{-2|\tau|}$.
   (a) Find $P(\{\mathbf{X}(t) \leq 3\})$.
   (b) Find $E[(\mathbf{X}(t+1) - \mathbf{X}(t-1))^2]$.

10. (*Papoulis* 9-10) Assume $\mathbf{X}(t)$ is a WSS, zero-mean, Gaussian random process. Show that if $\mathbf{Z}(t) = \mathbf{X}^2(t)$, then

$$C_{\mathbf{ZZ}}(\tau) = 2C^2_{\mathbf{XX}}(\tau).$$

11. (*Papoulis* 9-37) The random process $\mathbf{X}(t)$ is a WSS Gaussian random process with zero-mean and

$$R_{\mathbf{X}}(\tau) = Ie^{-\alpha|\tau|} \cos \beta\tau.$$

Show that if $\mathbf{Y}(t) = \mathbf{X}^2(t)$, then

$$C_{\mathbf{Y}}(\tau) = I^2 e^{-2\alpha|\tau|}(1 + 2\cos 2\beta\tau).$$

Find the power spectral density $S_{\mathbf{YY}}(\omega)$.

(Problems 12 and 13 have been removed from the assignment.)

14. Let $\mathbf{X}(t)$ be two real, independent, wide-sense stationary random processes defined on a random experiment. Let $\mathbf{Z}(t)$ be a new random process defined as

$$\mathbf{Z}(t) = \mathbf{X}(t)\mathbf{Y}(t).$$

Under what conditions is $\mathbf{Z}(t)$ wide-sense stationary (WSS)?

15. If the random process $\mathbf{X}(t)$ s given by

$$\mathbf{X}(t) = \cos(\omega_o t + \mathbf{\Theta}),$$

where $\mathbf{\Theta}$ is a random variable uniformly distributed on the interval $[0, 2\pi)$ and $\mathbf{Y}(t)$ is a wide-sense stationary random process with autocorrelation function

$$R_{\mathbf{Y}}(\tau) = e^{-\alpha|\tau|},$$

where α is a positive constant. Assume $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are statistically independent. Let

$$\mathbf{Z}(t) = \mathbf{X}(t)\mathbf{Y}(t).$$

Find the power spectral density of $\mathbf{Z}(t)$.