Homework Assignment #2
Should be completed by Session 5

Reading Assignment: Read Section 2-3 of Papoulis. Review all class notes.

1. (*Papoulis*, Problem 2-2) If $A = \{2 \leq x \leq 5\}$ and $B = \{3 \leq x \leq 6\}$, Find $A \cup B$, $A \cap B$, and $(A \cup B) \cap (\overline{A \cap B})$.
2. (*Papoulis*, Problem 2-3) Show that if $A \cap B = \emptyset$, then $P(A) \leq P(\overline{B})$.
3. (*Papoulis*, Problem 2-4) Show that
   (a) if $P(A) = P(B) = P(A \cap B)$, then $P((A \cap \overline{B}) \cup (B \cap \overline{A})) = 0$;
   (b) if $P(A) = P(B) = 1$, then $P(A \cap B) = 1$.
4. (*Papoulis,* Problem 2-5.) Prove and generalize the following identity:

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

   By generalize, we mean to the union of n events.
5. (*Papoulis,* Problem 2-6) Show that if sample space $\mathcal{S}$ of a random experiment consists of a countable number of outcomes ξ_i and each subset $\{\xi_i\}$ is an event in the event space, then every subset of $\mathcal{S}$ is an event in the event space of the random experiment.
6. (*Papoulis*, Problem 2-7) If $\mathcal{S} = \{1, 2, 3, 4\}$ is the sample space of a random experiment, find the smallest σ-field that contains the events $\{1\}$ and $\{2, 3\}$.
7. (*Papoulis*, Problem 2-8) If $A \subset B$, P(A)=1/4, and $P(B) = 1/3$, find $P(A|B)$ and $P(B|A)$.
8. (*Papoulis*, Problem 2-9) Show that

   $$P(A \cap B|C) = P(A|B \cap C)P(B|C)$$

   and

   $$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C).$$

9. Show that for any two events A and B in a probability space $(\mathcal{S}, \mathcal{F}, P)$ the following relationship holds:

   $$P(A)P(B) - P(A \cap B) = P(\overline{A} \cap B) - P(\overline{A})P(B) = P(A \cap \overline{B}) - P(A)P(\overline{B}).$$

10. Express each of the following events in terms of the events A, B, and C and the operations of complementation, union, and intersection:
    (a) at least one of the events A, B, C occurs;
    (b) at most one of the events A, B, C occurs;
    (c) none of the events A, B, C occurs;
    (d) aall three events occur;
    (e) exactly one of the events A, B, C occurs;
    (f) A and B occur, but not C;
    (g) A occurs, if not then B does not occur either.

11. Let $\mathcal{S}$ be the sample space corresponding to the random experiment of tossing a coin three times and noting the sequence of H and T (*heads* and *tails*). Let A be the event that heads occurs exactly twice, let B be the event that at least two heads appear, and let C be the event that heads appears when tails has appeared at least once.
    (a) Give the elements of A, B, and C;
    (b) Describe the events: (i) $\overline{A} \cap B$, (ii) $\overline{A} \cap \overline{B}$, (iii) $A \cap C$.