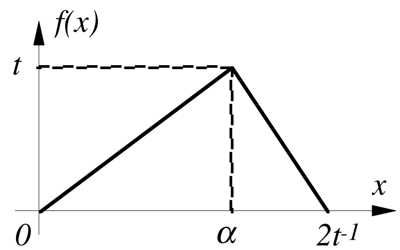
ECE642 Information Theory and Source Coding — Mark R. Bell
Fall 2023 — MSEE 336

Homework Assignment #4
Due November 10, 2023

1. An (n, k, t) binary channel block code is a binary block code that encodes k binary source digits into n binary codeword digits and for which decoder exists such that all patterns of t or less errors in the n-bit codeword can be correctly decoded as the transmitted n-bit codeword. Note that an (n, k, t) code may in general be able to correct some, but not all, error patterns of $t + 1$ errors, but in this problem we will consider only codes that can correct all t error patterns but no patterns of $t + 1$ or more errors. For each of the following binary codes, determine the rate and block error probability P_e of the code when transmitted over a binary symmetric channel with crossover probability ϵ, and plot P_e for $10^{-5} \leq \epsilon \leq 1$.
   a. $(5, 1, 2)$ Binary Repetition Code;
   b. $(7, 4, 1)$ Binary Hamming Code;
   c. $(23, 12, 3)$ Binary Golay Code.
2. One trivial method of constructing a binary channel code is by using the idea of a repetition code. A $(2t+1, 1, t)$ is a rate $1/(2t+1)$ binary channel code in which a single digit is encoded by repeating it $2t + 1$ times. There are only two codewords in this code, the "all ones" codeword $(1, 1, \ldots, 1)$ and the "all zeros" codeword $(0, 0, \ldots, 0)$. The decoding rule is to decode as the "majority vote." That is if there are more zeroes than ones received, decode as $(0, 0, \ldots, 0)$ transmitted; if there are more ones than zeros received, decode as $(1, 1, \ldots, 1)$. (a) Write an expression for the block error probability of the $(2t + 1, 1, t)$ binary repetition code. (b) A $(k(2t + 1), k, s)$ extended binary repetition code can be formed by concatenating k $(2t+1, 1, t)$ binary repetition codes to transmit k binary source digits using $k(2t + 1)$ codeword digits. What is the value of s for such a code? What is the block-error probability of the $(k(2t + 1), k, s)$ extended binary repetition code when used to transmit information over a *binary symmetric channel* with crossover probability ϵ? For $k = 4$, Plot this as a function of ϵ for $10^{-5} \leq \epsilon \leq 1$. How does this compare to the block-error probability for the $(7, 4, 1)$ Hamming code in Problem 1, part b. How do the rates for these two codes compare?
3. If X, Y, and Z are jointly-distributed random variables defined on the real line, verify the following properties of differential entropy:
   a. $h(XY) \leq h(X) + h(Y)$, with equality iff X and Y are statistically independent;
   b. $h(X) \geq h(X|Y)$, with equality iff X and Y are statistically independent;
   c. $h(YZ|X) \leq h(Y|X) + h(Z|X)$, and state the conditions under which equivalence holds.

4. Compute the differential entropies of the following probability densities:
   a. The exponential density: $f(x) = \lambda e^{-\lambda x}$, $X \geq 0$; (λ is a positive constant);
   b. The arcsine density: $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$, $0 < x < 1$;
   c. The triangle density, given as follows:



5. Find well defined density functions $f(x)$ (i.e., no delta functions or limits) such that
   a. $h(X) = +\infty$;
   b. $h(X) = -\infty$.
6. Find the density function $f(x)$ of the continuous random variable X defined on the real line such that $\mathrm{E}(X) = \mu$, $\mathrm{E}[(X-\mu)^2] = \sigma^2$, and $h(X)$ is maximum. What is this maximum value of $h(X)$. (*Hint:* there are at least two ways to work this problem: a good guess of $f(x)$ and application of Jensen's Inequality, or Lagrange multipliers.)
7. Let X be a random variable defined on the real line satisfying the following set of constraints:

$$\mathrm{E}\{g_i(X)\} = \int_{-\infty}^{\infty} g_i(x) f(x)\, dx = \eta_i, \quad i = 1, \ldots, n.$$

   Here, the n functions $g_i(x)$ are assumed to be known functions defined on the real line. Show that the density function $f(x)$ which maximizes $h(X)$ is given by

$$f(x) = A \exp\{-\lambda_1 g_1(x) - \cdots - \lambda_n g_n(x)\},$$

   where the λ_i are constants that can be determined from the constraint equations and A can be determined from the fact that $f(x)$ must be a valid pdf. Show that the resulting maximum differential entropy is

$$h(X) = \lambda_1 \eta_1 + \cdots + \lambda_n \eta_n.$$

8. Let X and Y be two jointly-distributed real random variables. Show that
   a. $I(X:Y) \geq 0$, with equality iff X and Y are statistically independent;
   b. $I(X;Y) = I(Y;X)$.
9. Using the result of problem 6 above, show that the cost-capacity function of the additive Gaussian Noise channel is

$$C(\beta) = \frac{1}{2} \log\left(1 + \frac{\beta}{\sigma_z^2}\right).$$