ECE600 Random Variables and Waveforms
Spring 2024

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MSEE 336

Homework Assignment #7
Should be completed by Session 21

Reading Assignment: Read Sections 6-4 through 6-7 of Papoulis.

1. If $\mathbf{X}$ and $\mathbf{Y}$ are zero-one random variables that indicate the events A and B respectively, *i.e.,*
$$\mathbf{X} = 1_A(\omega)$$
and
$$\mathbf{Y} = 1_B(\omega),$$
where $A, B \in \mathcal{F}$ of the probability space $(\mathcal{S}, \mathcal{F}, P)$, then
   (a) find the probability masses in the (x, y)-plane;
   (b) show that the random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent if and only if the events A and B are statistically independent.
2. (*Papoulis* 6-15) The random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent and $\mathbf{Y}$ is uniformly distributed on the interval $(0, 1)$. Show that if
$$\mathbf{Z} = \mathbf{X} + \mathbf{Y},$$
then
$$f_{\mathbf{Z}}(z) = F_{\mathbf{X}}(z) - F_{\mathbf{X}}(z-1).$$
3. (*Papoulis* 6-16) (a) The function $g(x)$ is monotone increasing and $\mathbf{Y} = g(\mathbf{X})$. Show that
$$F_{\mathbf{XY}}(x, y) = \begin{cases} F_{\mathbf{X}}(x), & \text{if } y > g(x), \\ F_{\mathbf{Y}}(y), & \text{if } y < g(x). \end{cases}$$
   (b) Find $F_{\mathbf{XY}}(x, y)$ if $g(x)$ is monotone decreasing.
4. (*Papoulis* 6-24) Express $F_{\mathbf{ZW}}(z, w)$ in terms of $F_{\mathbf{XY}}(x, y)$ if $\mathbf{Z} = \max(\mathbf{X}, \mathbf{Y})$ and $\mathbf{W} = \min(\mathbf{X}, \mathbf{Y})$.
5. (*Papoulis* 6-18) The random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent with
$$f_{\mathbf{X}}(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \cdot 1_{[0,\infty)}(x)$$
and
$$f_{\mathbf{Y}}(y) = \begin{cases} 1/\pi\sqrt{1-y^2}, & \text{for } |y| \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$
Show that the random variable $\mathbf{Z} = \mathbf{XY}$ is Gaussian with mean 0 and variance α^2.

6. (*Papoulis* 6-19) The random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent Rayleigh random variables with p.d.f.s

$$f_{\mathbf{X}}(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \cdot 1_{[0,\infty)}(x) \qquad \text{and} \qquad f_{\mathbf{Y}}(y) = \frac{y}{\beta^2} e^{-y^2/2\beta^2} \cdot 1_{[0,\infty)}(y).$$

(a) Show that if $\mathbf{Z} = \mathbf{X}/\mathbf{Y}$, then

$$f_{\mathbf{Z}}(z) = \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \cdot 1_{[0,\infty)}(z).$$

(b) Using the result of part (a), show that for any $k > 0$,

$$P(\{\mathbf{X} \le k\mathbf{Y}\}) = \frac{k^2}{k^2 + \alpha^2/\beta^2}.$$

7. Consider two jointly distributed random variables $\mathbf{X}$ and $\mathbf{Y}$ having joint p.d.f.

$$f_{\mathbf{XY}}(x,y) = \begin{cases} 1/\pi, & \text{for } x^2 + y^2 \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal p.d.f. of $\mathbf{X}$.
(b) Find the p.d.f. of the new random variable $\mathbf{R} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2}$.
(c) Are $\mathbf{X}$ and $\mathbf{Y}$ statistically independent? Justify your answer (*i.e.*, show whether or not they are independent.)
(d) Are $\mathbf{X}$ and $\mathbf{Y}$ uncorrelated? Justify your answer (*i.e.*, show whether or not they are uncorrelated.)