ECE600 Random Variables and Waveforms
Spring 2018

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MSEE 336

Midterm Exam #1
Session 9
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75 minutes

1	25
2	25
3	25
4	25
Total	100

Name: Solutions (M.R.B.)

ID #:

Directions:

1. **Print** your name on all test pages.
2. Exam is closed book, closed notes, and no calculators.
3. Problems have designated weights.
4. Clearly designate all answers asked for (arrows, underline, box, etc.)

1. **(25 pts.)** *This problem consists of two separate short questions relating to the structure of probability spaces:*

   (a) Assume that $\mathcal{S}$ is the sample space of a random experiment and that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-fields (valid event spaces) on $\mathcal{S}$. Show that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-field on $\mathcal{S}$.

   (b) Consider a sample space $\mathcal{S}$ and corresponding event space $\mathcal{F}$. Suppose that P_1 and P_2 are both valid probability measures defined on $\mathcal{F}$. Show that P defined by

$$P(A) = \alpha_1 P_1(A) + \alpha_2 P_2(A), \quad \forall A \in \mathcal{F}$$

   is also a valid probability measure on $\mathcal{F}$ if $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

(a) If $\mathcal{F}$ is a σ-field, then we know it is a non-empty collection of subsets of $\mathcal{S}$ satisfying

(i) If $A \in \mathcal{F}$, then $\overline{A} \in \mathcal{F}$;

(ii) If $A_1, \ldots, A_n \in \mathcal{F}$, then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F};$$

(iii) If $A_1, \ldots, A_n, \ldots \in \mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

We now check that these 3 closure properties are satisfied for $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$:

Clearly $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is not empty, since $\phi, \mathcal{S} \in \mathcal{F}_1$ and $\phi, \mathcal{S} \in \mathcal{F}_2 \Rightarrow \phi, \mathcal{S} \in \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$.

(i): $A \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow \begin{cases} A \in \mathcal{F}_1 \Rightarrow \overline{A} \in \mathcal{F}_1, \text{ because } \mathcal{F}_1 \text{ is a } \sigma\text{-field}, \\ A \in \mathcal{F}_2 \Rightarrow \overline{A} \in \mathcal{F}_2, \text{ because } \mathcal{F}_2 \text{ is a } \sigma\text{-field}, \end{cases}$

$\Rightarrow \overline{A} \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}.$

(ii) If $A_1, \ldots, A_n \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow \begin{cases} \bigcup_{i=1}^{n} A_i \in \mathcal{F}_1, & \mathcal{F}_1 \text{ is a } \sigma\text{-field} \\ \bigcup_{i=1}^{n} A_i \in \mathcal{F}_2, & \mathcal{F}_2 \text{ is a } \sigma\text{-field} \end{cases}$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}$

(iii) If $A_1, \ldots, A_n, \ldots \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow \begin{cases} \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1, & \mathcal{F}_1 \text{ is a } \sigma\text{-field} \\ \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_2, & \mathcal{F}_2 \text{ is a } \sigma\text{-field.} \end{cases}$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}.$

So $\mathcal{F} \cap \mathcal{F}_2$ is a non-empty collection of subsets of $\mathcal{S}$ satisfying the closure axioms of a σ-field. $\therefore$ $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field on $\mathcal{S}$.

(Problem 1 Solution Continued)

(b) $\mathcal{P} = \alpha_1 \mathcal{P}_1 + \alpha_2 \mathcal{P}_2$ is a valid probability measure if it satisfies the axioms of probability:

(i) $\mathcal{P}(A) \geq 0, \forall A \in \mathcal{F}$;

(ii) $\mathcal{P}(\mathcal{S}) = 1$;

(iii) If $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then

$$\mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}(A_i);$$

(iv) If $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then

$$\mathcal{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathcal{P}(A_i).$$

Let's check these for $\mathcal{P} = \alpha_1 \mathcal{P}_1 + \alpha_2 \mathcal{P}_2$, $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$:

(i): $\mathcal{P}(A) = \alpha_1 \mathcal{P}_1(A) + \alpha_2 \mathcal{P}_2(A) \geq 0, \forall A \in \mathcal{F}$, because $\alpha_1, \alpha_2 \geq 0$ and $\mathcal{P}_1(A), \mathcal{P}_2(A) \geq 0, \forall A \in \mathcal{F}$, because $\mathcal{P}_1$ and $\mathcal{P}_2$ are both probability measures.

$\therefore$ Axiom (i) is satisfied

(ii):

$$\begin{aligned} \mathcal{P}(\mathcal{S}) &= \alpha_1 \mathcal{P}_1(\mathcal{S}) + \alpha_2 \mathcal{P}_2(\mathcal{S}) \\ &= \alpha_1 \cdot 1 + \alpha_2 \cdot 1, \quad \text{because } \mathcal{P}_1(\mathcal{S}) = 1 \text{ and } \mathcal{P}_2(\mathcal{S}) = 1 \\ &= \alpha_1 + \alpha_2 = 1 \quad \therefore \text{ Axiom (ii) is satisfied.} \end{aligned}$$

(iii): If $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint, then

$$\mathcal{P}_1\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}_1(A_i) \quad \text{and} \quad \mathcal{P}_2\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathcal{P}_2(A_i).$$

Thus it follows that

$$\begin{aligned} \mathcal{P}\left(\bigcup_{i=1}^{n} A_i\right) &= \alpha_1 \mathcal{P}_1\left(\bigcup_{i=1}^{n} A_i\right) + \alpha_2 \mathcal{P}_2\left(\bigcup_{i=1}^{n} A_i\right) \\ &= \alpha_1 \sum_{i=1}^{n} \mathcal{P}_1(A_i) + \alpha_2 \sum_{i=1}^{n} \mathcal{P}_2(A_i) \\ &= \sum_{i=1}^{n} \left(\alpha_1 \mathcal{P}_1(A_i) + \alpha_2 \mathcal{P}_2(A_i)\right) = \sum_{i=1}^{n} \mathcal{P}(A_i) \end{aligned}$$

$\therefore$ Axiom (iii) is satisfied.

(Problem 1 Solution Continued)

(iv):
$$\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \alpha_1 P_1\left(\bigcup_{i=1}^{\infty} A_i\right) + \alpha_2 P_2\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \alpha_1 \sum_{i=1}^{\infty} P_1(A_i) + \alpha_2 \sum_{i=1}^{\infty} P_2(A_i) \\
&= \sum_{i=1}^{\infty} \left(\alpha_1 P_1(A_i) + \alpha_2 P_2(A_i)\right) \\
&= \sum_{i=1}^{\infty} P(A_i).
\end{aligned}$$

$\therefore$ Axiom (iv) is satisfied

$\therefore$ $P = \alpha_1 P_1 + \alpha_2 P_2$ satisfies the axioms of probability and is hence a valid probability measure.

2. **(25 pts.)** **(25 pts.)** Consider the following probability mass functions (pmfs) with sample spaces $\mathcal{S}$ as specified:

Binomial: $\mathcal{S} = \{0, 1, \ldots, n\}$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n, \quad 0 \le p \le 1;$$

Poisson: $\mathcal{S} = \{0, 1, 2, \ldots\}$

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots, \quad \lambda > 0;$$

Geometric: $\mathcal{S} = \{1, 2, 3 \ldots\}$

$$p(k) = C\alpha^k, \quad k = 1, 2, 3, \ldots, \quad 0 < \alpha < 1,$$

and C is an appropriately chosen constant.

(a) List the conditions necessary for an arbitrary $p(k)$ to be a valid pmf for a sample space $\mathcal{S}$.

(b) Show that the Binomial pmf specified above is a valid pmf.

(c) Show that the Poisson pmf specified above is a valid pmf.

(d) Determine the value of C (in terms of α) that make the Geometric pdf specified above a valid pmf.

(a) In order for $p(k)$ to be a valid probability mass function (pmf) for sample space $\mathcal{S}$, it must satisfy the following two conditions:

(i) $p(k) \ge 0, \ \forall k \in \mathcal{S}$;

(ii) $\displaystyle\sum_{k \in \mathcal{S}} p(k) = 1$.

(b) In order to show that the binomial pmf is a valid pmf, we must show that it satisfies conditions (i) and (ii) in part (a):

(i): $p(k) = \binom{n}{k} p^k (1-p)^{n-k} = \underbrace{\frac{n!}{k!(n-k)!}}_{>0} \cdot \underbrace{p^k}_{\ge 0} \cdot \underbrace{(1-p)^{n-k}}_{\ge 0} \ge 0.$

(ii): $\displaystyle\sum_{k \in \mathcal{S}} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \overset{\text{Bin. Thm}}{=} (p + (1-p))^n = 1^n = 1.$

$\therefore$ The binomial p.m.f is a valid pmf.

(Problem 2 Solution Continued)

(c) In order to show the Poisson pmf is a valid pmf, we must show that it satisfies conditions (i) and (ii) in part (a):

(i): $p(k) = \frac{\lambda^k e^{-\lambda}}{k!} = \underbrace{\frac{1}{k!}}_{>0} \cdot \underbrace{\lambda^k}_{>0} \cdot \underbrace{e^{-\lambda}}_{>0} \geq 0.$

(ii): $\sum_{k \in \mathcal{S}} p(k) = \sum_{k=0}^{\infty} p(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$= e^{-\lambda} e^{\lambda}$, because $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$ (Taylor series expansion of e^{λ} about $\lambda = 0$.)

$= e^0 = 1.$

$\therefore$ The Poisson pmf is a valid pmf.

(d) $p(k) = C\alpha^k \geq 0$ for any $C \geq 0$ (given that $\alpha > 0$.)

In order to find the value of C that makes $p(k)$ a valid pmf, we must find C such that $\sum_{k=1}^{\infty} p(k) = 1.$

$$\sum_{k \in \mathcal{S}} p(k) = \sum_{k=1}^{\infty} p(k) = \sum_{k=1}^{\infty} C\alpha^k = C\alpha \sum_{k=1}^{\infty} \alpha^{k-1} = C\alpha \sum_{m=0}^{\infty} \alpha^m$$

$= C\alpha \cdot \frac{1}{1-\alpha}$, since $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$, $|r| < 1$

$$\Rightarrow \frac{C\alpha}{1-\alpha} = 1 \Rightarrow \boxed{C = \frac{1-\alpha}{\alpha}}$$

n.b. Because $0 < \alpha < 1$, $C > 0$, so this C makes $p(k)$ a valid pmf, and

$$p(k) = (1-\alpha)\alpha^{k-1}, \quad k = 1, 2, 3, \ldots$$

3. **(25 pts.)** A survey is being carried out to determine what fraction of Purdue students use Purdue computers for personal (non-academic) purposes. Because some students would be uncomfortable admitting to using Purdue computers for non-academic purposes, each person questioned is asked to roll a fair die in secret and answer NO if the number "1" is shown, YES if the number "6" is shown, but to truthfully answer YES or NO if "2", "3", "4", or "5" are shown on the die. Because the outcome of the die is not revealed, it is impossible to tell from the answer given whether or not a student actually uses Purdue computers for non-academic reasons.

Suppose that the probability that a student answers YES in this survey is found to be 2/3.

(a) What is the true probability that a student uses Purdue computers for non-academic reasons?

(b) How likely is it that a student who answered YES in the survey actually uses Purdue computers for non-academic reasons?

(c) How likely is it that a student who answered NO in the survey actually uses Purdue computers for non-academic reasons?

(a) Let C = person questioned uses computers for non-academic purposes.

Y = person questioned answered YES

D_1 = die shows "1"

D_6 = die shows "6"

D_0 = die shows "2", "3", "4", or "5".

According to the rules of the survey

$$P(Y|D_1) = 0, \quad P(Y|D_6) = 1, \quad \text{and} \quad P(Y|D_0) = P(C).$$

Because $\{D_1, D_6, D_0\}$ is a partition of S, by the total probability law, we have

$$\begin{aligned}
P(Y) &= P(Y \cap S) = P(Y \cap (D_1 \cup D_6 \cup D_0)) \\
&= P((Y \cap D_1) \cup (Y \cap D_6) \cup (Y \cap D_0)) \\
&= P(Y \cap D_1) + P(Y \cap D_6) + P(Y \cap D_0) \\
&= P(Y|D_1)P(D_1) + P(Y|D_6)P(D_6) + P(Y|D_0)P(D_0) \\
&= 0 \cdot \tfrac{1}{6} + 1 \cdot \tfrac{1}{6} + P(C) \cdot \tfrac{4}{6} = \tfrac{2}{3}P(C) + \tfrac{1}{6}.
\end{aligned}$$

(Problem 3 Solution Continued)

So if $P(Y) = \frac{2}{3}$, we have

$$P(C) = \frac{3}{2}\left(P(Y) - \frac{1}{6}\right) = \frac{3}{2}\left(\frac{2}{3} - \frac{1}{6}\right) = 1 - \frac{3}{12} = \frac{9}{12} = \boxed{\frac{3}{4}}$$

(b) $$P(C|Y) = \frac{P(C \cap Y)}{P(Y)} = \frac{P(Y|C)P(C)}{P(Y)}$$

$$= \frac{(5/6)(3/4)}{(2/3)} = \frac{5 \cdot 3}{2 \cdot 2 \cdot 4} = \boxed{\frac{15}{16}}$$

(c) $$P(C|\bar{Y}) = \frac{P(C \cap \bar{Y})}{P(\bar{Y})} = \frac{P(\bar{Y}|C)P(C)}{P(\bar{Y})}$$

$$= \frac{(1 - P(Y|C))P(C)}{1 - P(Y)} = \frac{(1 - 5/6)(3/4)}{1 - 2/3}$$

$$= \frac{(\frac{1}{6})(\frac{3}{4})}{(1/3)} = \frac{9}{24} = \boxed{\frac{3}{8}}$$

4. **(25 pts.)** Three cups each contain red and blue marbles. *Cup A* contains one red and one blue marble. *Cup B* contains two red and one blue marble. *Cup C* contains three red and one blue marble. A cup is selected at random, and then a marble is selected at random from the selected cup.
   (a) What is the probability a red marble is selected?
   (b) Given that a red marble was selected, what is the probability that *Cup A* was the selected cup?

(a) $P(R) = P(R \cap S) = P(R \cap (A \cup B \cup C))$

where $\{A, B, C\}$ is a partition of S.

$= P(\underbrace{(R \cap A) \cup (R \cap B) \cup (R \cap C)}_{\text{union of disjoint events}})$, because $\cap$ is distributive over $\cup$.

$= P(R \cap A) + P(R \cap B) + P(R \cap C)$, by axiom 3

$= P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C)$, by defn. of conditional prob.

$= \frac{1}{2} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3}$

$= \frac{1}{6} + \frac{2}{9} + \frac{1}{4} = \frac{6+8+9}{36} = \boxed{\frac{23}{36}}$

(b) $P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{P(R|A)P(A)}{P(R)}$

$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{23}{36}} = \frac{1/6}{23/36} = \boxed{\frac{6}{23}}$