Session 10

Random Variables

"A function of the outcome of a random experiment taking on a numerical value."

Mappings and Functions:

Defn: Given two abstract spaces $\mathcal{S}$ and $\mathcal{A}$, an $\mathcal{A}$-valued function mapping $\mathcal{S}$ to $\mathcal{A}$

$$f : \mathcal{S} \to \mathcal{A}$$

is an assignment of a specific element of $\mathcal{A}$ to each $\omega \in \mathcal{S}$.

Thus, given $\omega \in \mathcal{S}$, $f(\omega) \in \mathcal{A}$, $\forall \omega \in \mathcal{S}$.

Given

$$f : \mathcal{S} \to \mathcal{A}$$

$\mathcal{S}$ is called the <u>domain</u> of f;

$\mathcal{A}$ is called the <u>range</u> of f.

<u>Example</u>: In calculus we study real valued functions of a real variable:

$$f : \mathbb{R} \to \mathbb{R}.$$

$$\mathcal{S} = \mathbb{R} \quad \text{and} \quad \mathcal{A} = \mathbb{R}.$$

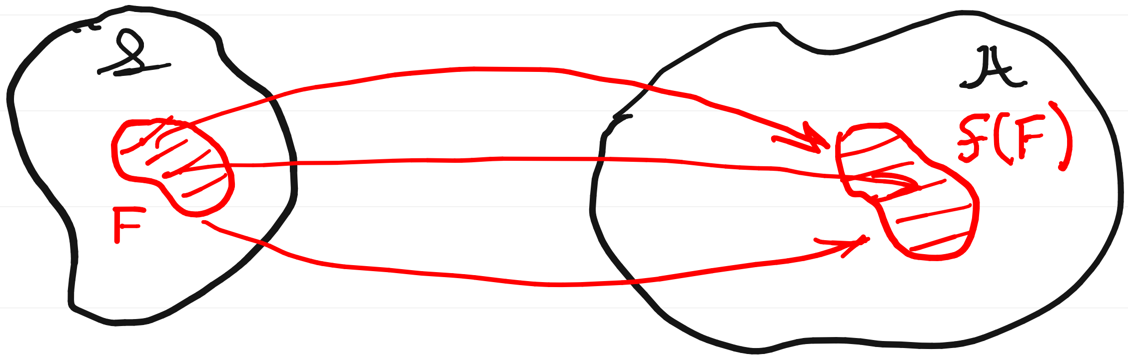
Defn: Given any two sets $F \subset \mathcal{S}$ and $G \subset \mathcal{A}$, we define the <u>image</u> of F under f as

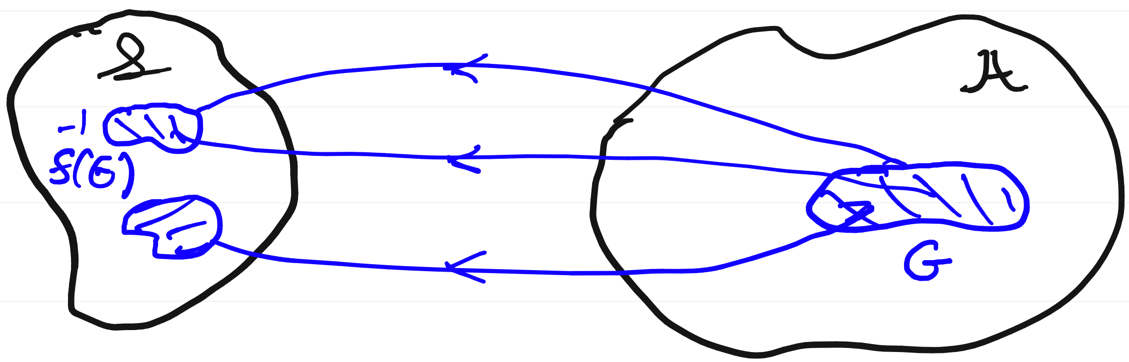
$$\text{“}f(F)\text{”} \triangleq \{a \in \mathcal{A} : a = f(w) \text{ for some } w \in F\}.$$

and the <u>pre-image</u> or <u>inverse image</u> of $G \subset \mathcal{A}$ under f is defined as

$$\text{“}f^{-1}(G)\text{”} \triangleq \{w \in \mathcal{S} : f(w) \in G\}$$

Pictorially, the situation appears as follows:





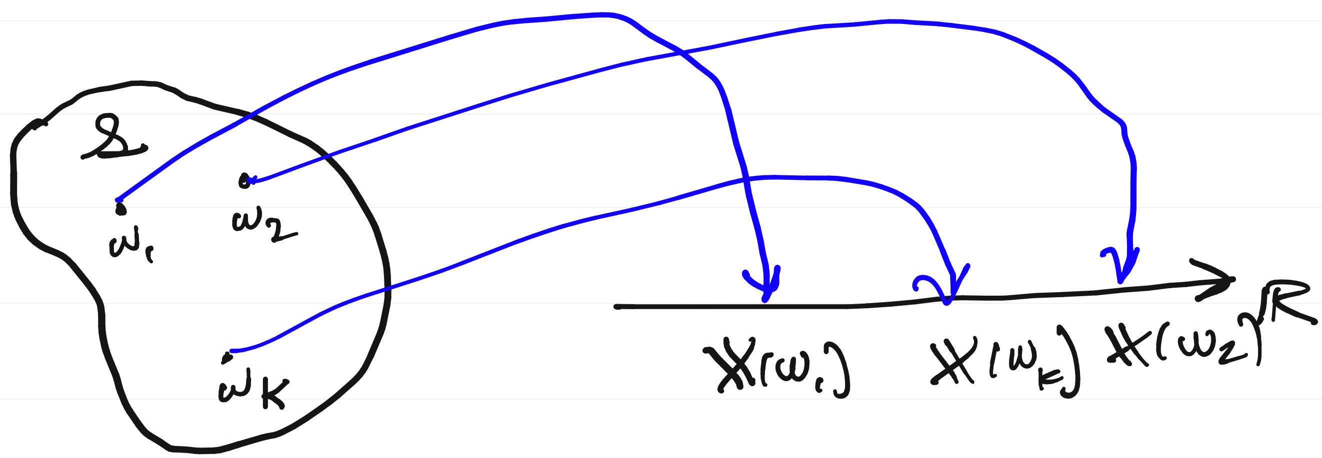
Random Variables

- We often characterize the outcome of a random experiment by a number (perhaps a measurement.)

e.g.
- Induced electric current in an antenna due to random thermal motion of charges.
- The number of "Heads" that occur in N independent coin tosses.

<u>Intuitive "Defn":</u> Given $(S, \mathcal{F}, P)$, a <u>random variable</u> is a mapping from S to the real line.

$$X : S \to \mathbb{R}$$



A random variable is a real-valued function defined on $\mathcal{S}$

- $\mathcal{S}$ is its domain
- $\mathbb{R}$ is its range

1. It is <u>not</u> random.
2. It is <u>not</u> a variable.

The mapping $X : \mathcal{S} \to \mathbb{R}$ is fixed and $\underset{\text{(non-random)}}{\underline{\text{deterministic}}}$

The randomness observed in $X(\omega)$ is due to the randomness of the outcome $\omega \in \mathcal{S}$ in the random experiment $(\mathcal{S}, \mathcal{F}, P)$

A function $\mathbb{X} : \mathcal{S} \to \mathbb{R}$ has the property that its output inherits a probability measure from P in the underlying probability space $(\mathcal{S}, \mathcal{F}, P)$:

We require that $\mathbb{X}(\cdot)$ has the property that for any $A \in \mathcal{B}(\mathbb{R})$, we can compute the probability $P_X(A)$.

Any function $\mathbb{X}(\cdot)$ such that we can compute $P_X(A)$ for all $A \in \mathcal{B}(\mathbb{R})$ is called a <u>(Borel) measurable function.</u>

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$ a random variable is a mapping $\mathbb{X} : \mathcal{S} \to \mathbb{R}$ with the property that for all $A \in B(\mathbb{R})$,

$$\mathbb{X}^{-1}(A) = \{ w \in \mathcal{S} : \mathbb{X}(w) \in A \} \in \mathcal{F}$$

Such a function is called a Borel measurable function.

(Borel) Measurability of X insures that we can compute the probability that

$$\text{“}\{X \in A\}\text{”} = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$$

for all $A \in \mathcal{B}(\mathbb{R})$.

Q: How restrictive is measurability on the family of functions that are valid RVs?

e.g. If $(\mathcal{S}, \mathcal{F}, P)$ is taken to have

$$\mathcal{S} = \mathbb{R}$$
$$\mathcal{F} = \mathcal{B}(\mathbb{R})$$

Then almost any function

$$X : \mathbb{R} \to \mathbb{R}$$

will be a random variable.

In fact, for

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), P) \xRightarrow{X(\cdot)} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

all of the following will be valid RVs:

1. Continuous functions
2. Polynomials
3. Step functions: $k \cdot u_{-1}(x-a)$
4. All indicator functions:

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

for all $A \in \mathcal{B}(\mathbb{R})$.

5. All trig. functions (circular and hyperbolic)
6. All Bessel functions

7. Limits of sequences of measurable function

$$f_1(x),\ f_2(x),\ \dots,\ f_n(x)\ \dots$$

$$\lim_{n\to\infty} f_n(x) = f(x)$$

8. Sums (finite and countable) of measurable functions

$$f_1(x),\ f_2(x),\ \dots,\ f_n(x),\ \dots :$$

$$f(x) = \sum_n f_n(x).$$

Fact: It is very difficult to describe a function $X(\cdot)$ that is not measurable.

e.g. If A is one of the "problem sets" in $\mathcal{P}(\mathbb{R}) - \mathcal{B}(\mathbb{R})$, then

$$X(\omega) = 1_A(\omega)$$

is not measurable and hence not a random variable.

The "Range Space" of a function f

Let $f : \mathcal{S} \to \mathcal{A}$.

We call the image $f(\mathcal{S}) \subset A$ the <u>range space</u> of f.

<u>e.g.</u> $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

then "$f(\mathbb{R})$" $= [0, \infty) =$ range space of $f(x) = x^2$.

Given $(\mathcal{S}, \mathcal{F}, P)$ and a R.V X, the R.V. takes on values in the range space

$$\mathcal{C} = f(\mathcal{S}) \subset \mathbb{R}.$$