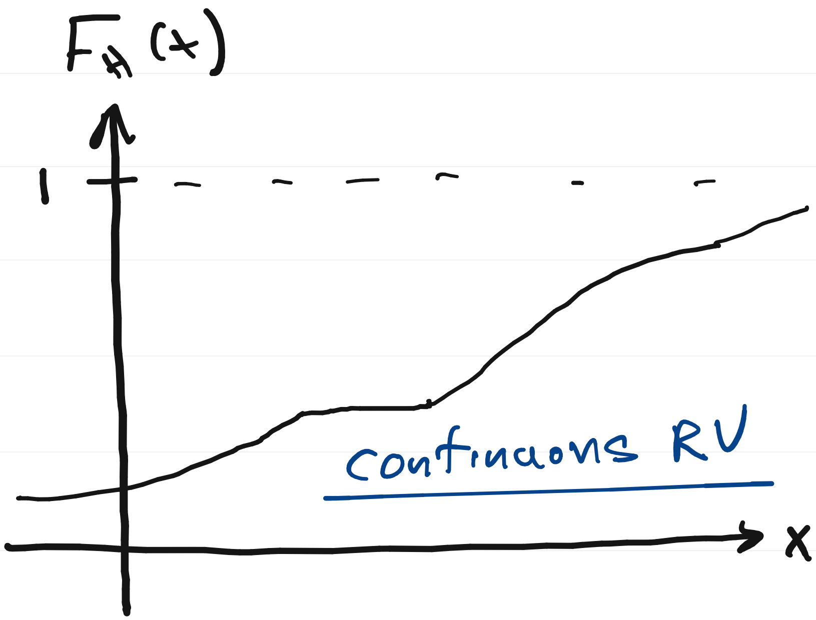
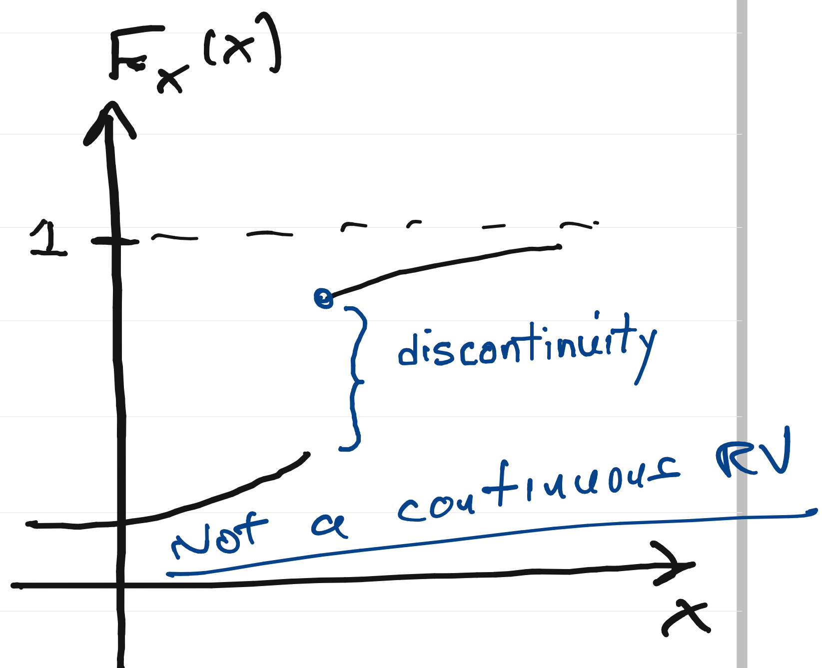
Session 12

Recall...

12.1

Defn: We say that a random variable is (absolutely) continuous if $F_X(x)$ is a continuous function at all points $x \in \mathbb{R}$.



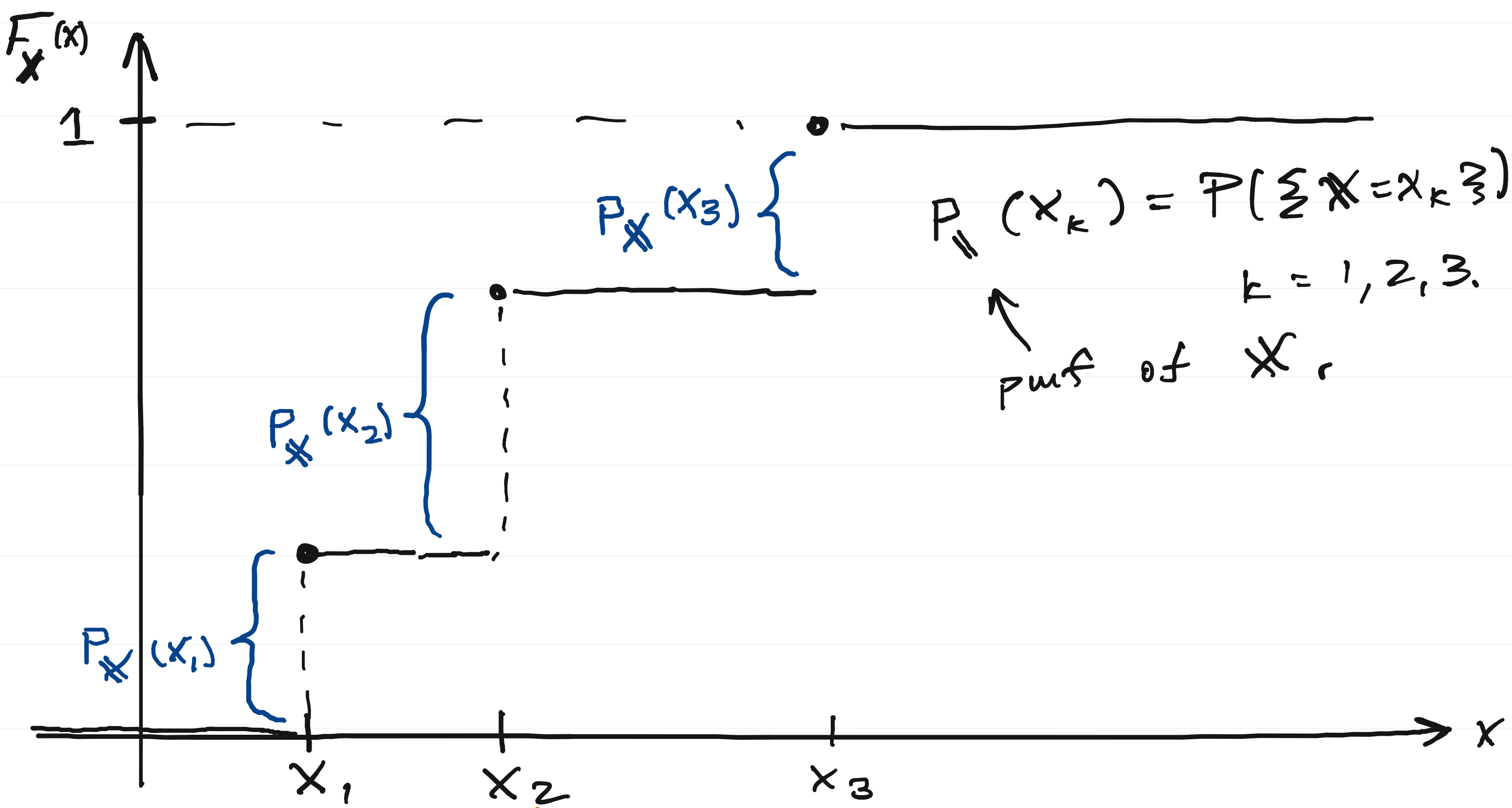


Recall...

12.2

We say that a RV X is <u>discrete</u> if it takes on values from a discrete (finite or countable) subset of $\mathbb{R}$.

In this case, $F_X(x)$ is a "staircase function".



$P_X(x_k) = P(\{X = x_k\})$

$k = 1, 2, 3.$

pmf of X.

Defn: The probability density function of a RV $\mathbb{X}$ is defined as the derivative of the cdf $F_{\mathbb{X}}(x)$ w.r.t. x:

$$f_{\mathbb{X}}(x) = \frac{dF_{\mathbb{X}}(x)}{dx}.$$

n.b. (i) $f_{\mathbb{X}}(x) \geq 0, \quad \forall x \in \mathbb{R}.$

(ii) $$\int_{-\infty}^{\infty} f_{\mathbb{X}}(x)\, dx = F_{\mathbb{X}}(\infty) - F_{\mathbb{X}}(-\infty)$$
$$= 1 - 0$$
$$= 1.$$

n.b. We will broaden our "definition" of derivative to include the derivative of a step discontinuities as Dirac δ-functions.

Dirac δ-functions : $\delta(x)$

(i) $\delta(x) = 0 \quad , \quad \forall x \neq 0.$

$\left[\delta(x - x_0) = 0 \quad , \quad \forall x \neq x_0 \right)$

(ii) $$\int_{-\infty}^{\infty} \delta(x)\, dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx = 1 \quad , \quad \forall \varepsilon > 0$$

These two defining properties give rise to the "sifting property" of the Dirac delta function:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = \int_{-\infty}^{\infty} g(x_0)\,\delta(x-x_0)\,dx$$

$$= g(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = g(x_0)\cdot 1$$

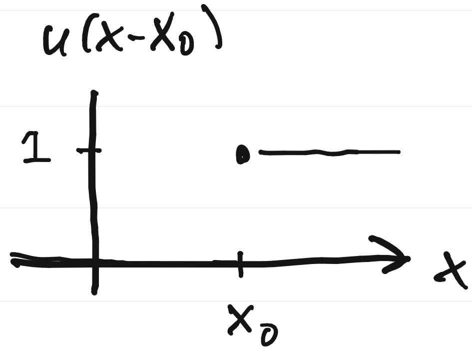
$$= g(x_0)$$

$$\therefore \boxed{\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)}$$

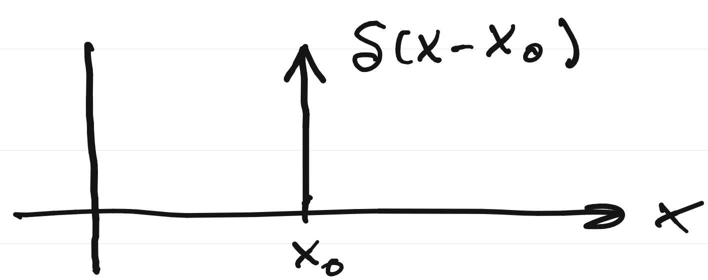
The sifting property of the Dirac δ-fcn.

Suppose I have

$$u(x-x_0) \triangleq 1_{[x_0,\infty)}(x) = \begin{cases} 1, & x \in [x_0,\infty) \\ 0, & x \in (-\infty, x_0) \end{cases}$$



$$\xRightarrow{\frac{d}{dx}}$$



$$\text{"}\frac{d}{dx} u(x-x_0)\text{"} = \delta(x-x_0)$$

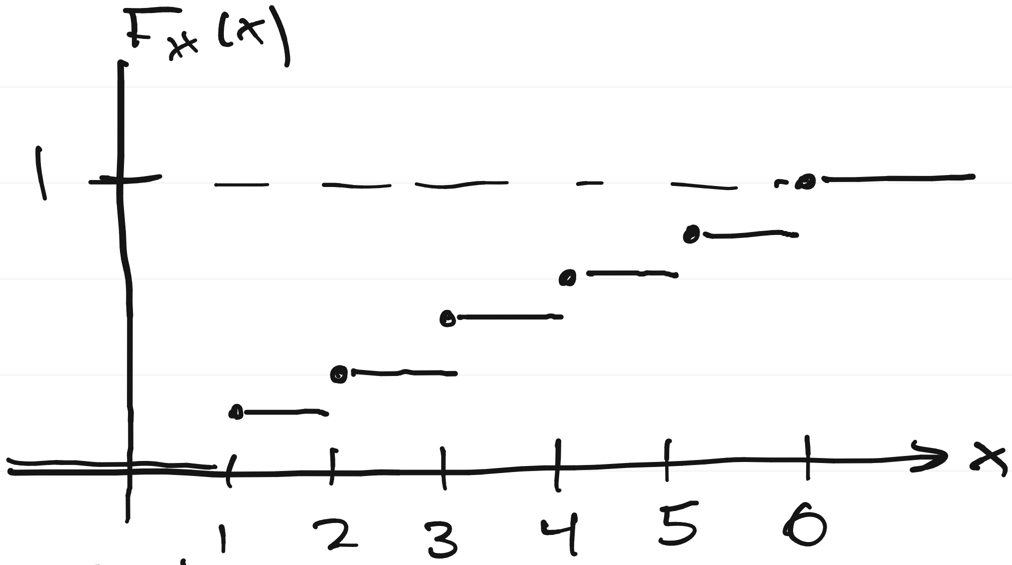
$$v(x) = \int_{-\infty}^{x} \delta(r-x_0)\,dr = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \\ 1, & x = x_0 \end{cases}$$

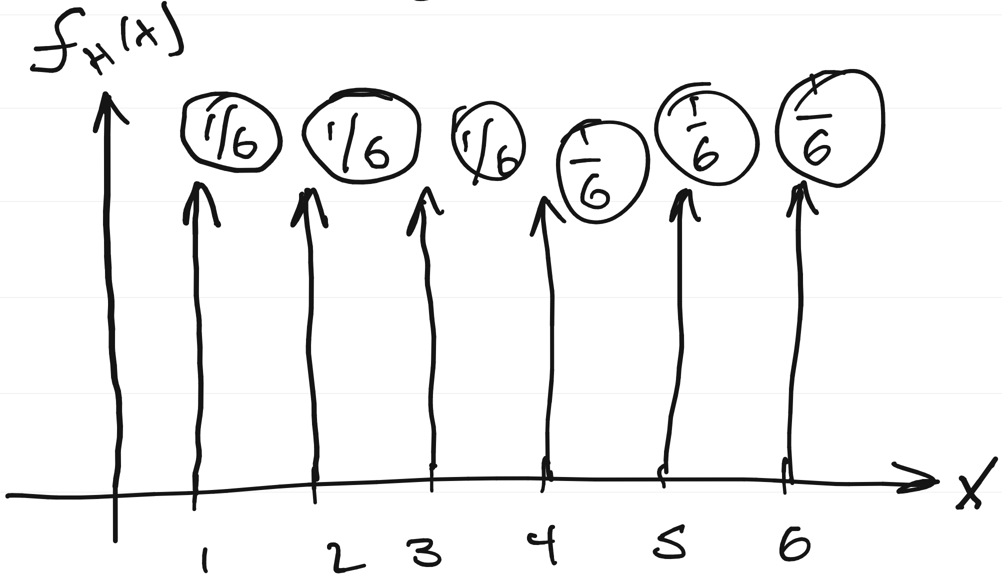
Example: Consider a RV that is the numerical outcome of rolling a fair die.

$$F_X(x) = P(\{X \leq x\})$$

$$= \frac{1}{6} \cdot \mathbb{1}_{[1,\infty)}(x) + \frac{1}{6}\mathbb{1}_{[2,\infty)}(x) + \cdots + \frac{1}{6}\mathbb{1}_{[6,\infty)}(x)$$

$$f_X(x) = \text{"}\frac{dF_X(x)}{dx}\text{"} = \frac{1}{6}\delta(x-1) + \frac{1}{6}\delta(x-2) + \cdots + \frac{1}{6}\delta(x-6)$$


$F_X(x)$
1
x
1
2
3
4
5
6


$f_X(x)$
1/6
1/6
1/6
1/6
1/6
1/6
X
1
2
3
4
5
6

Properties of the pdf of a RV

1. $f_X(x) \geq 0 \quad , \forall x \in \mathbb{R}$

2. $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$

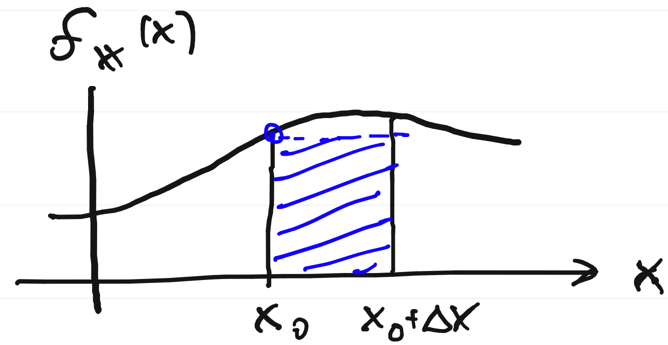
3. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

4. $P(\{x_1 < X \leq x_2\}) = \int_{x_1}^{x_2} f_X(x)\, dx$

For a continuous RV $\mathbb{X}$

$$P(\{x_0 < \mathbb{X} \le x_0 + \Delta x\}) = \int_{x_0}^{x_0+\Delta x} f_{\mathbb{X}}(x)\,dx$$

$$\simeq f_{\mathbb{X}}(x_0)\cdot \Delta x \quad , \quad \text{for small } \Delta x$$



Recall: $\quad f_{\mathbb{X}}(x) \cong \dfrac{F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)}{\Delta x}$

$$\Rightarrow \quad f_{\mathbb{X}}(x)\cdot \Delta x \cong F_{\mathbb{X}}(x+\Delta x) - F_{\mathbb{X}}(x)$$

$$= P(\{x < \mathbb{X} \le x_0 + \Delta x\})$$

We often describe a RV X by specifying its cdf or pdf and completely ignoring the underlying $(\mathcal{S}, \mathcal{F}, P)$.

(The underlying $(\mathcal{S}, \mathcal{F}, P)$ is there, but we just don't think about it.)

Ex. 1 Gaussian RV

A RV X is Gaussian if it has a pdf of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad \forall x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

n.b. $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \underbrace{\Phi\left(\frac{x-\mu}{\sigma}\right)}$ $\left(G\left(\frac{x-\mu}{\sigma}\right)\text{ in Papoulis}\right)$

$\mathcal{N}[\mu, \sigma^2]$

where $\Phi(r) = \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$.

$\Phi(\cdot)$ cannot be written in "closed form." It can be numerically computed and is widely tabulated.

So if X is a Gaussian RV with $\mu \in \mathbb{R}$ and $\sigma > 0$, then

$$P(\{a < X \leq b\}) = F_X(b) - F_X(a)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

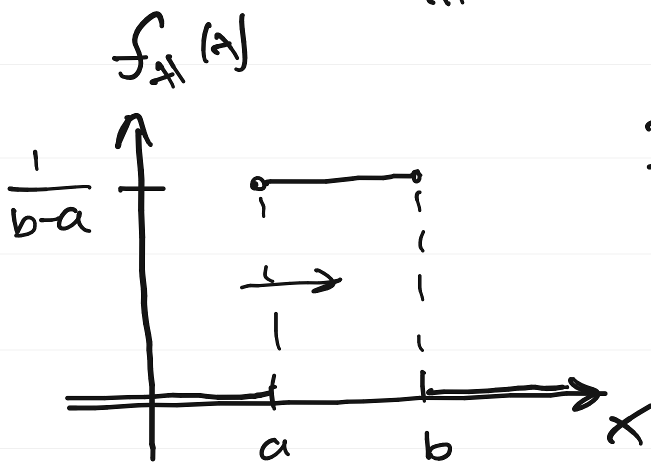
Ex. 2 Uniformly Distributed RV

A RV has a uniform distribution,

$$X \sim U[a, b], \quad a < b$$

if

$$f_X(x) = \frac{1}{b-a} \cdot 1_{[a,b]}(x)$$



Integrate $\Rightarrow$