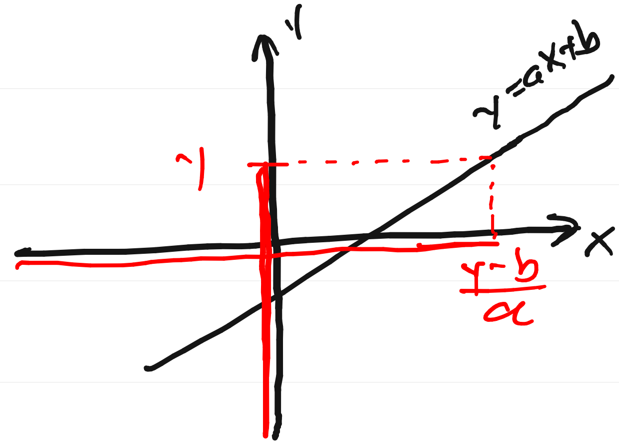
Session 15

Recall...

Ex. A $\quad Y = aX + b, \quad a, b \in \mathbb{R}$ 15.1

$\Rightarrow \quad g(x) = ax + b$ (affine transformation)

two cases: $\quad a \gtrless 0$.



(i) $a > 0$

$$F_Y(y) = P(\{Y \le y\}) = P(\{aX + b\} \le y\})$$
$$= P(\{X \le \tfrac{y-b}{a}\})$$
$$= F_X\left(\tfrac{y-b}{a}\right)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$
$$= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Recall...

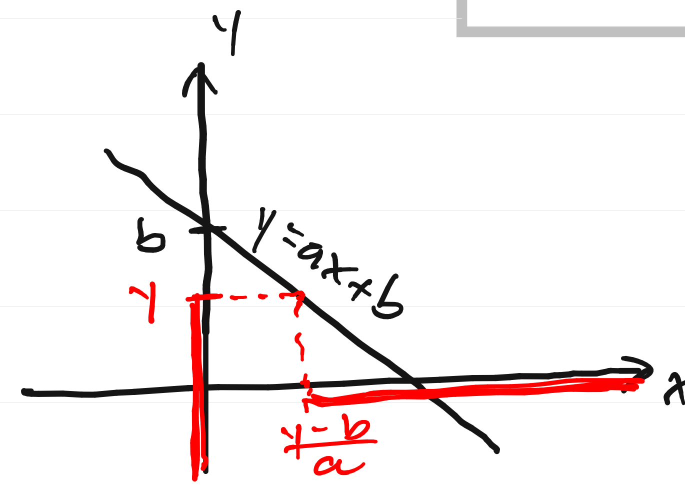
(ii) $\underline{a<0}$: $F_Y(y) = P(\{Y \le y\})$

$$= P(\{aX+b \le y\})$$

$$= P\left(\left\{X \ge \frac{y-b}{a}\right\}\right)$$

$$= P\left(\left\{X > \frac{y-b}{a}\right\}\right)$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$



$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[1 - F_X\left(\frac{y-b}{a}\right)\right]$$

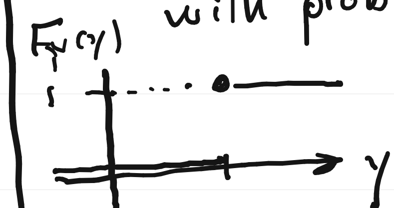
$$= -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

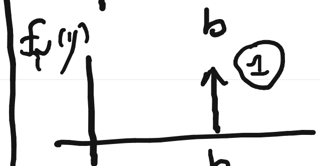
$$= \frac{-1}{a} f_X\left(\frac{y-b}{a}\right)$$

combining (i) and (ii):

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

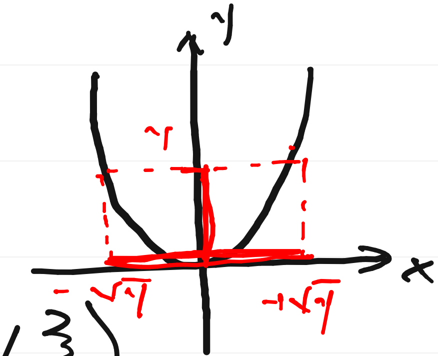
$\underline{a=0}$: $Y = b$ with prob. 1





Ex. B $\quad \underline{Y} = g(X) = X^2 \Rightarrow g(x) = x^2.$

Note immediately that $F_Y(y) = 0, \; \underline{y < 0.}$



For $y > 0$:

$$F_Y(y) = P(\{Y \le y\}) = P(\{X^2 \le y\}).$$

$$= P(\{-\sqrt{y} \le X \le \sqrt{y}\})$$

$$= P(\{-\sqrt{y} \le X \le \sqrt{y}\})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\therefore F_Y(y) = \left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] \cdot 1_{(0,\infty)}(y)$$

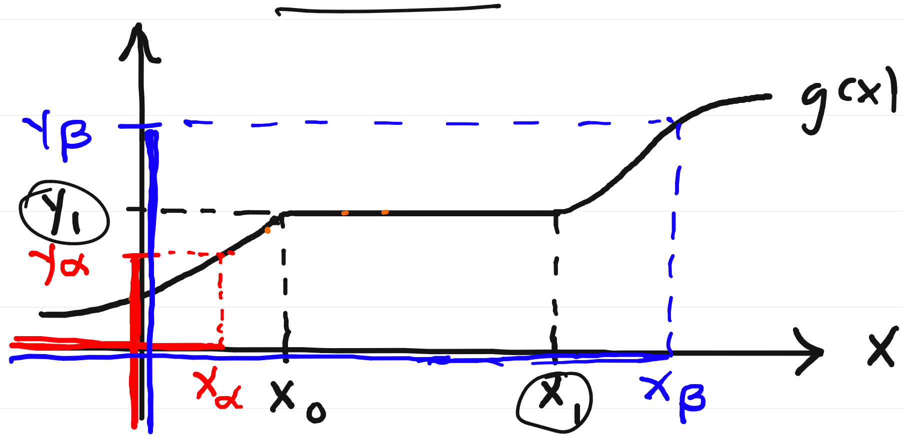
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\left(\frac{-1}{2\sqrt{y}}\right)$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], \; y > 0$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \cdot 1_{(0,\infty)}(y).$$

Ex. 2 Suppose that $g(x)$ is constant across an interval $[x_0, x_1]$



$$F_Y(y) = P(\{Y \le y\})$$

(i) $F_Y(y_\alpha) = P(\{Y \le y_\alpha\})$

$$= P(\{X \le x_\alpha\})$$

$$= P(\{X \le \text{"}g^{-1}(y_\alpha)\text{"}\})$$

$$= P(\{X \in g^{-1}(-\infty, y_\alpha]\})$$

$$= F_X(\text{"}g^{-1}(y_\alpha)\text{"})$$

(ii) $y > y_1$: $F_Y(y_\beta) = P(\{Y \le y_\beta\}) = P(\{X \le x_\beta\})$

$$= P(\{X \le \text{"}g^{-1}(y_\beta)\text{"}\})$$

$$= F_X(g^{-1}(y_\beta))$$

15.6

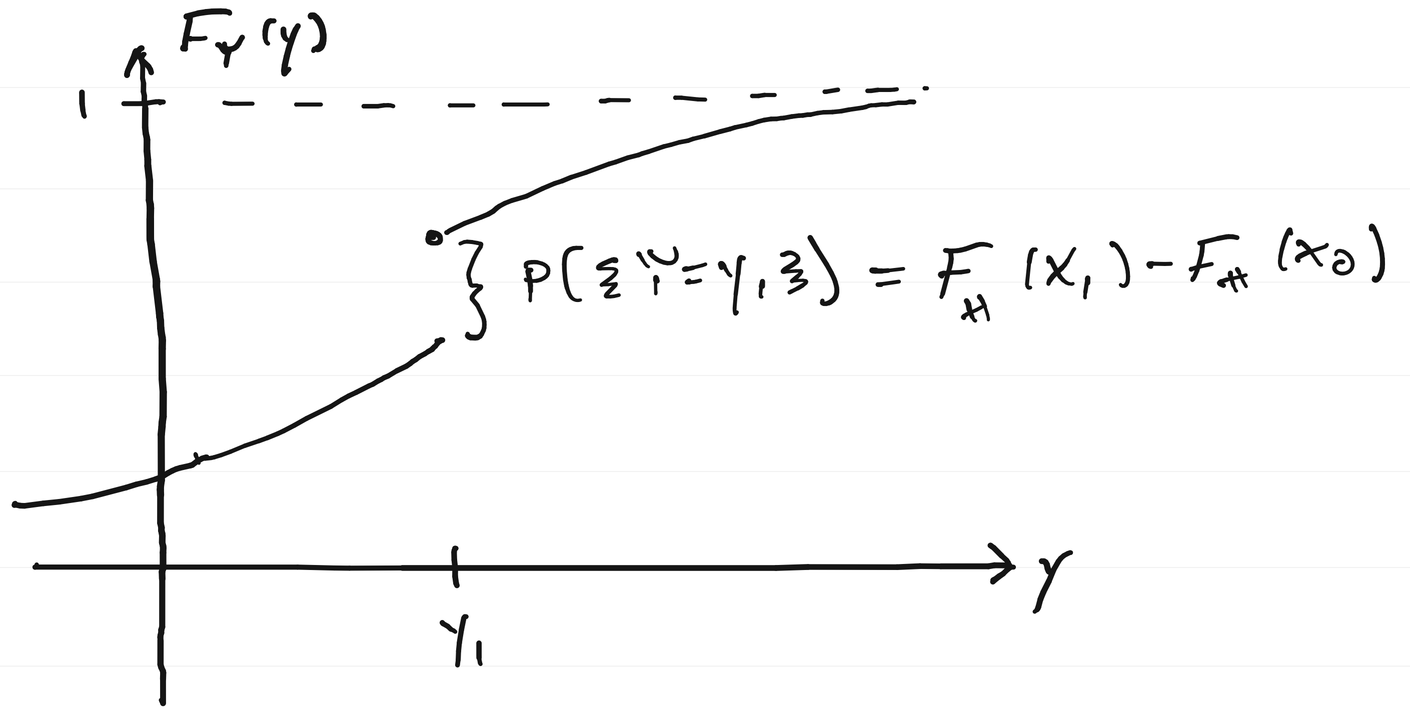
$$F_{\mathbb{Y}}(y_1) = P(\{\mathbb{Y} \le y_1\}) = P(\{\mathbb{X} \le x_1\})$$

$$= P(\{\mathbb{X} \le x_0\}) + P(\{x_0 < \mathbb{X} \le x_1\})$$

$$= F_{\mathbb{X}}(x_0) + \underbrace{\left[F_{\mathbb{X}}(x_1) - F_{\mathbb{X}}(x_0)\right]}_{\ge 0}$$

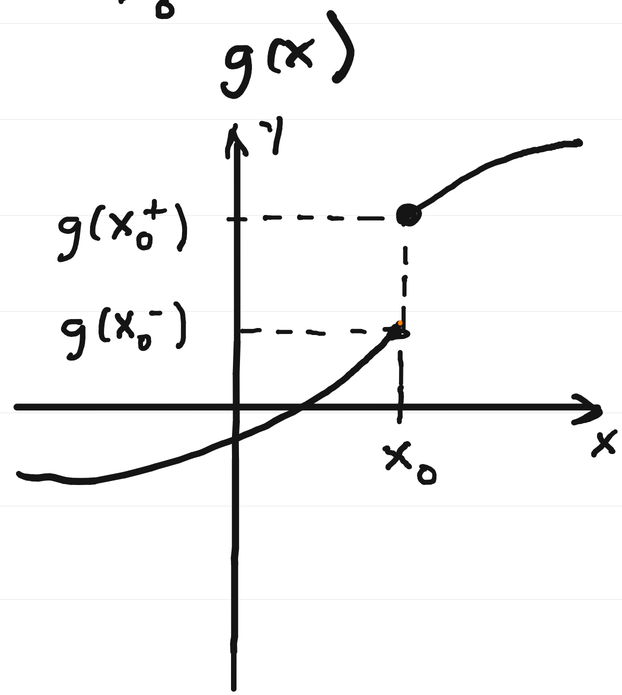
$$= F_{\mathbb{X}}(x_1)$$

$$\therefore \; F_Y(y) = \begin{cases} F_X(g^{-1}(y)), & y < y_1 \\ F_X(x_1), & y = y_1 \\ F_X(g^{-1}(y)), & y > y_1 \end{cases}$$



<u>Ex. 3</u> Assume that $g(x)$ has a jump discontinuity at x_0

$$g(x_0^+) \neq g(x_0^-)$$



Assume $g(x) < g(x_0^-), x < x_0$

$g(x) > g(x_0^+), x > x_0$

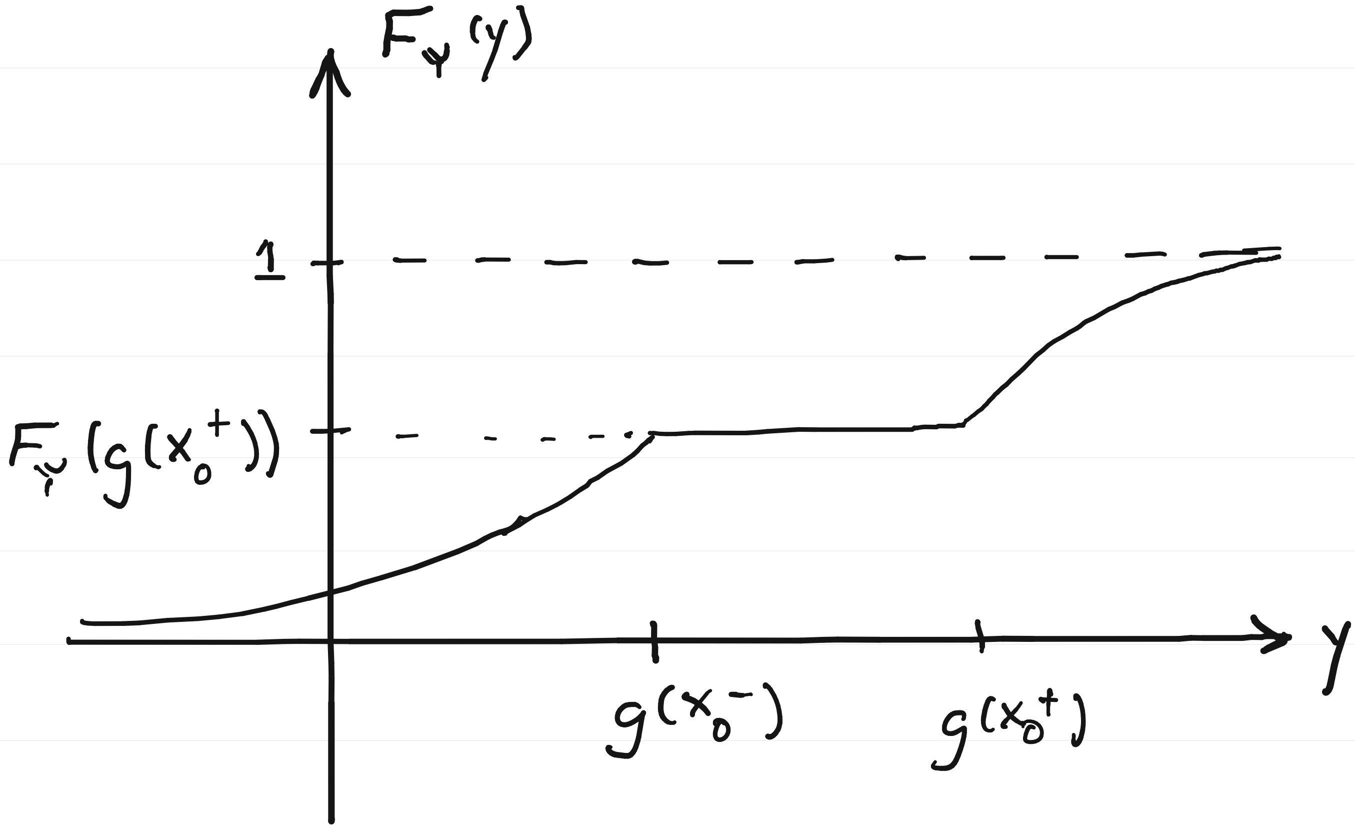
If y: $g(x_0^-) \leq y \leq g(x_0^+)$:

$$F_{\underline{y}}(y) = P(\{\underline{y} \leq y\}) = P(\{\underline{x} \leq x_0^+\}) = P(\{\underline{x} \leq x_0^-\})$$

$$= F_{\underline{x}}(x_0^+) = F_{\underline{x}}(x_0^-)$$

if $\underline{x}$ is absolutely continuous.

$\therefore F_Y(y)$ appears as follows:



The Direct pdf Method

Suppose $Y = g(X)$, where $g: \mathbb{R} \to \mathbb{R}$ such that the inverse $g^{-1}(\cdot)$ exists,

(i.e., is $y = g(x) \Rightarrow x = g^{-1}(y)$ is unique.),

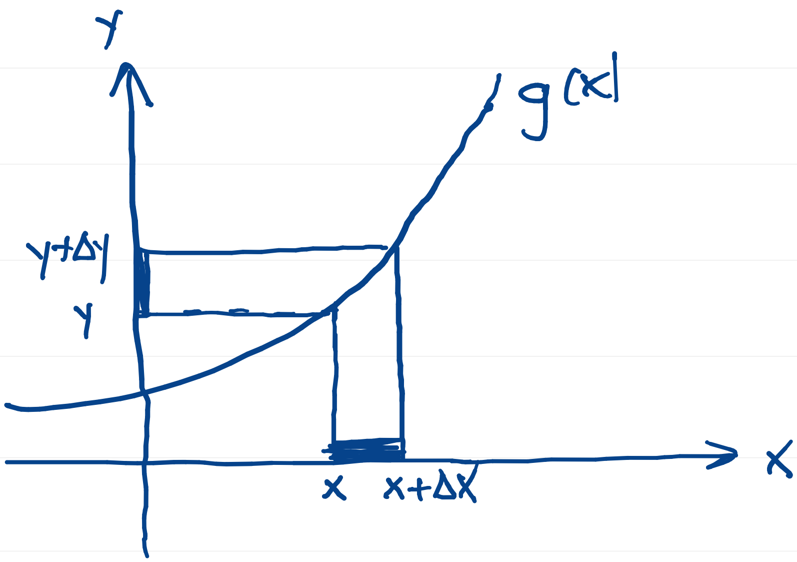
and assume that $x(y) = g^{-1}(y)$

$$\frac{dx}{dy} = \frac{dg^{-1}(y)}{dy} \text{ exists}$$

Then

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$= f_X(x(y)) \cdot \left| \frac{dx(y)}{dy} \right|, \quad \text{where } x(y) = g^{-1}(y).$$



$$P(\{X \in [x, x+\Delta x)\}) = P(\{Y \in [y, y+\Delta y]\})$$

$$\begin{array}{ccc} \wr\!\wr & & \wr\!\wr \\ f_X(x)\cdot\Delta x & = & f_Y(y)\cdot\Delta y \end{array}$$

$$\Rightarrow \quad f_Y(y) = f_X(x)\cdot\frac{\Delta x}{\Delta y} \simeq f_X(x)\,\frac{dx}{dy}.$$

Example: Let $X \sim U[0,1]$ and let

$$Y = g(X) = \sqrt{X}.$$

Find $f_Y(y)$.

$$f_X(x) = 1_{[0,1]}(x)$$

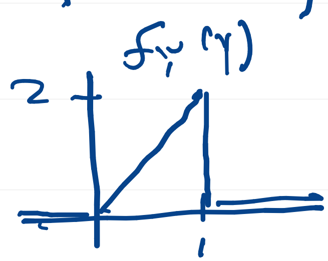
Solution: $y = g(x) = \sqrt{x} \Rightarrow x = y^2$

$$x(y) = y^2$$

$$\frac{dx(y)}{dy} = \frac{dy^2}{dy} \quad 2y$$

$$\Rightarrow \quad f_Y(y) = f_X(x(y)) \cdot |2y|$$

$$= 1_{[0,1]}(y^2) \cdot 2y = 2y \cdot 1_{[0,1]}(y^2)$$



Example: Let X be a Gaussian RV with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let $Y = aX + b$. Find $f_Y(y)$.

Solution: $Y = aX + b$

$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right|$$

$y = ax + b$

$x = \frac{y-b}{a}$

$x(y) = \frac{y-b}{a}$

$$\frac{dx}{dy} = \frac{1}{a}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(y-b)^2}{2a^2} \right\} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi}\,|a|} \exp\left\{ -\frac{(y-b)^2}{2a^2} \right\}.$$

Mean, Variance and Expectation

Defn: The mean or expected value of a RV X with pdf $f_X(x)$ is

$$E[X] \triangleq \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

What about discrete RVs.

<u>n.b.</u> The definition above also applies to discrete RVs if we write their pdf using δ-functions:

If $P(\{X = x_k\}) = p_X(x_k) = p_k$

over a discrete index set

$$f_X(x) = \sum_k p_X(x_k)\,\delta(x - x_k) = \sum_k p_k\,\delta(x - x_k)$$

and

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{-\infty}^{\infty} x \left(\sum_k p_k\,\delta(x - x_k) \right) dx$$

$$= \sum_k p_k \underbrace{\int_{-\infty}^{\infty} x\,\delta(x - x_k)\,dx}_{x_k} = \sum_k p_k x_k$$

$$= \sum_k p_X(x_k) \cdot x_k$$

$\therefore$ For a discrete RV X, we have

$$\boxed{E[X] = \sum_k x_k \, p_X(x_k).}$$

If you know about Riemann-Stiltjes integrals, you can write

$$E[X] = \int_{-\infty}^{\infty} x \, dF_X(x)$$

$$= \begin{cases} \sum_k x_k \, p_X(x_k) & (\text{discrete RV } X) \\ \int_{-\infty}^{\infty} x \, f_X(x) \, dx & (\text{continuous RV } X) \end{cases}$$

Defn: Let $\mathbb{X}$ be a RV on $(\mathcal{S}, \mathcal{F}, P)$ and let $M \in \mathcal{F}$. Then the conditional mean of $\mathbb{X}$ conditioned on M is

$$E[\mathbb{X} \mid M] \triangleq \int_{-\infty}^{\infty} x \, f_{\mathbb{X}}(x \mid M) \, dx.$$

(n.b. If $\mathbb{X}$ is discrete, we have the conditional pmf $p_{\mathbb{X}}(x_k \mid M) = P(\{\mathbb{X} = x_k\} \mid M)$, and then

$$E[\mathbb{X} \mid M] = \int_{-\infty}^{\infty} x f_{\mathbb{X}}(x \mid M) \, dx = \int_{-\infty}^{\infty} x \left(\sum_k p_{\mathbb{X}}(x_k \mid M) \, \delta(x - x_k) \right) dx$$

$$= \sum_k p_{\mathbb{X}}(x_k \mid M) \cdot \int_{-\infty}^{\infty} x \, \delta(x - x_k) \, dx = \sum_k x_k \, p_{\mathbb{X}}(x_k \mid M) \Big)$$

<u>Example</u>: Let $\mathbb{X}$ be an exponentially distributed RV with pdf

$$f_{\mathbb{X}}(x) = \frac{1}{\mu} \exp\left\{-\frac{x}{\mu}\right\} \mathbb{1}_{[0,\infty)}(x) \quad , \mu > 0$$

What is $E[\mathbb{X}]$?

$$E[\mathbb{X}] = \int_{-\infty}^{\infty} x \, f_{\mathbb{X}}(x) \, dx = \int_{0}^{\infty} x \cdot \frac{1}{\mu} e^{-x/\mu} \, dx$$

$$= \cdots = \left[-x e^{-x/\mu} - \mu e^{-x/\mu}\right]_0^{\infty} = \boxed{\mu}$$