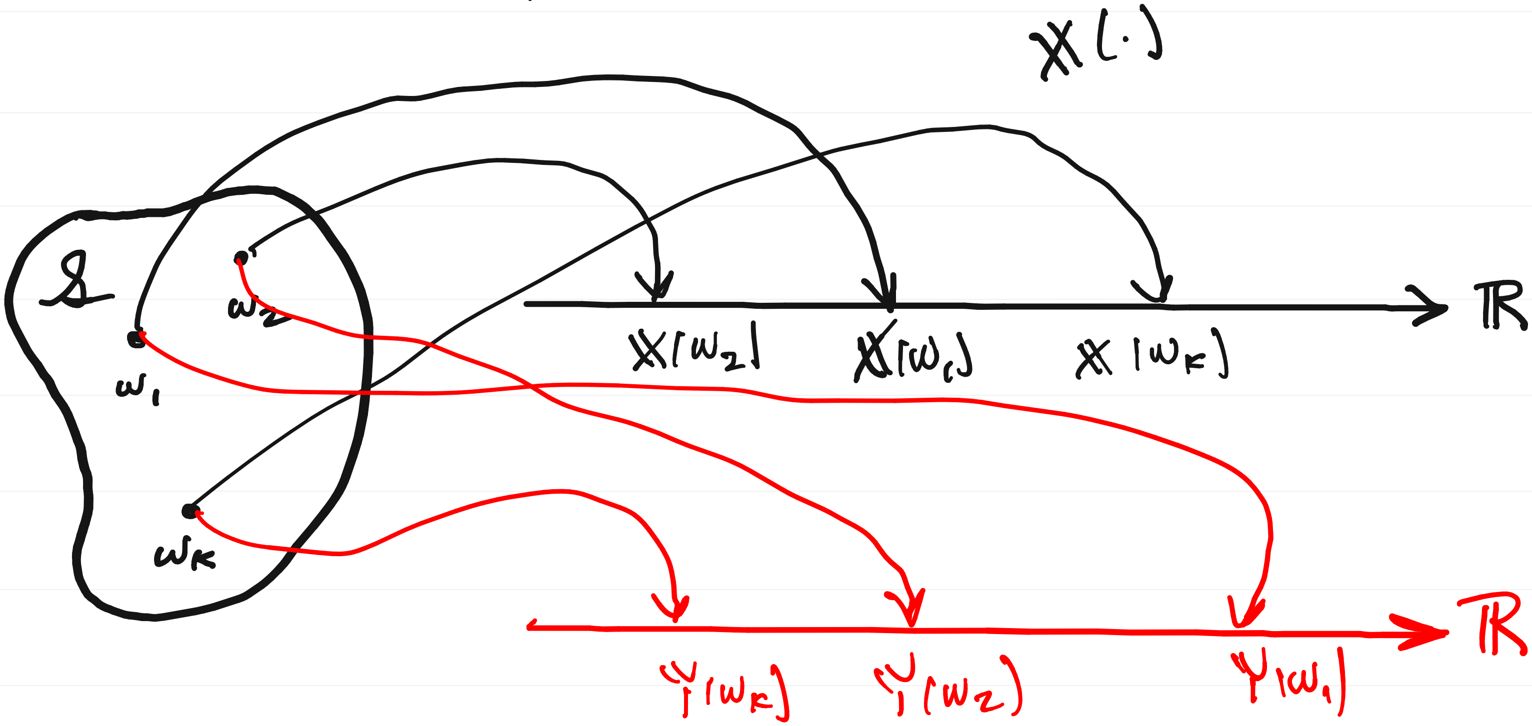
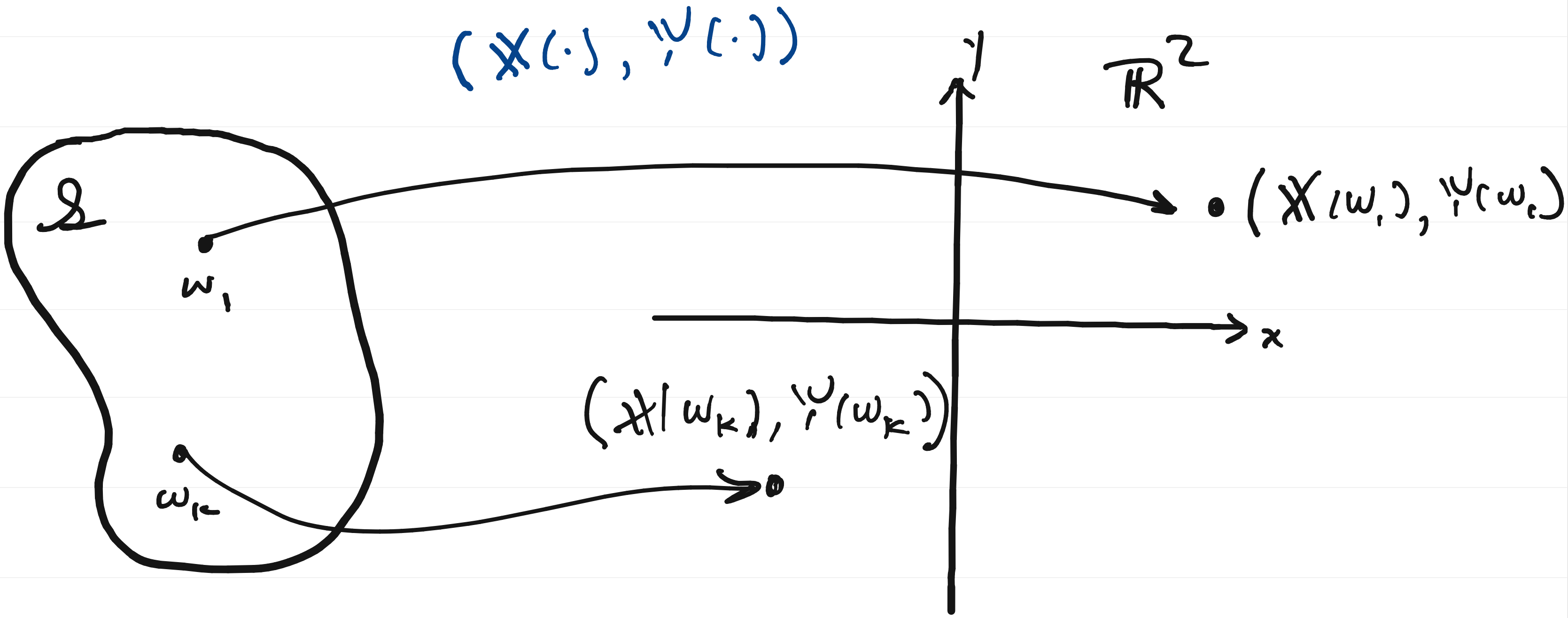
Session 19

Two Random Variables on $(\mathcal{S}, \mathcal{F}, P)$

If we can have one RV defined on $(\mathcal{S}, \mathcal{F}, P)$, why not two?



We can think of a pair of RVs on $(\mathcal{S}, \mathcal{F}, P)$ as mapping $\mathcal{S}$ to a point in the plane $\mathbb{R}^2$:



$$(X(\cdot), Y(\cdot)) : \mathcal{S} \to \mathbb{R}^2$$

Complex Random Variable

Given a pair of real RVs $\mathbb{X}$ and $\mathbb{Y}$ defined on $(\mathcal{S}, \mathcal{F}, P)$, we can define a complex RV as follows:

$$\mathbb{X}(\cdot) : \mathcal{S} \to \mathbb{R}, \quad \leftarrow \text{ Real part}$$

$$\mathbb{Y}(\cdot) : \mathcal{S} \to \mathbb{R}. \quad \leftarrow \text{ Imag part}$$

Define a complex RV $\mathbb{Z}$ as

$$\mathbb{Z} = \mathbb{X} + i\mathbb{Y} \qquad (\mathbb{Z}(\cdot) = \mathbb{X}(\cdot) + i\,\mathbb{Y}(\cdot)),$$

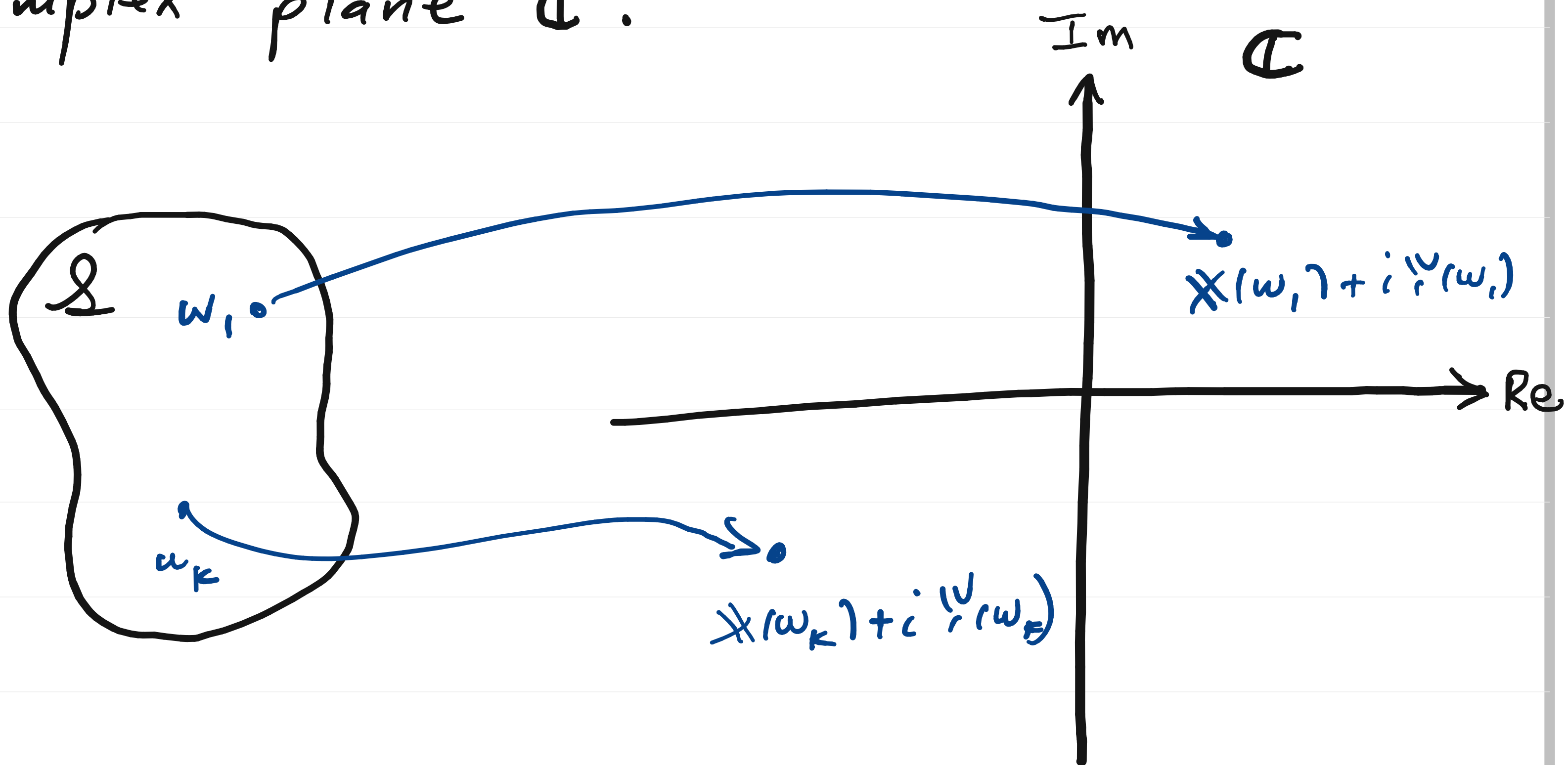
Then $\mathbb{Z} : \mathcal{S} \to \mathbb{C}$

$$\mathrm{Re}\{\mathbb{Z}\} = \mathbb{X}$$

$$\mathrm{Im}\{\mathbb{Z}\} = \mathbb{Y}$$

$$E[\mathbb{Z}] = E[\mathbb{X} + i\mathbb{Y}] = E[\mathbb{X}] + i\,E[\mathbb{Y}].$$

We can think of a pair of RVs on $(\mathcal{S}, \mathcal{F}, P)$ as mapping $\mathcal{S}$ to the complex plane $\mathbb{C}$:

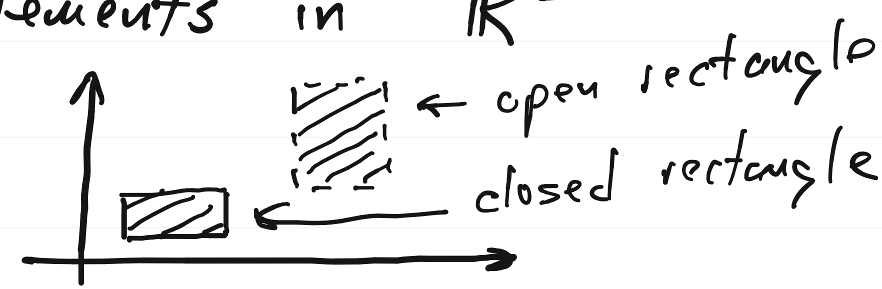


$$Z(\cdot) = X(\cdot) + iY(\cdot) : \mathcal{S} \to \mathbb{C}.$$

- Although we know $F_X(x)$ and $F_Y(y)$ fully describe the probabilistic behavior of X and Y separately, they do not (in general) characterize the joint probabilistic behavior of X and Y.

- Consider the set $D \subset \mathbb{R}^2$. $D \in \mathcal{B}(\mathbb{R}^2)$
  We will assume that D can be written as a countable union of open rectangles and their complements in $\mathbb{R}^2$

<u>Open Rectangle:</u>



$$D \in \mathcal{B}(\mathbb{R}^2) = \left\{ \begin{array}{l} \text{The smallest } \sigma\text{-field} \\ \text{containing all open} \\ \text{rectangles in } \mathbb{R}^2. \end{array} \right.$$

$$\mathcal{B}(\mathbb{R}^2) = \sigma\left(\{ \text{all open rectangles} \} \right)$$

We would like to compute the probability of the event

$$\{(X, Y) \in D\} = \{\omega \in S : (X(\omega), Y(\omega)) \in D\}.$$

Knowing $F_X(x)$ and $F_Y(y)$ is not sufficient to do this.

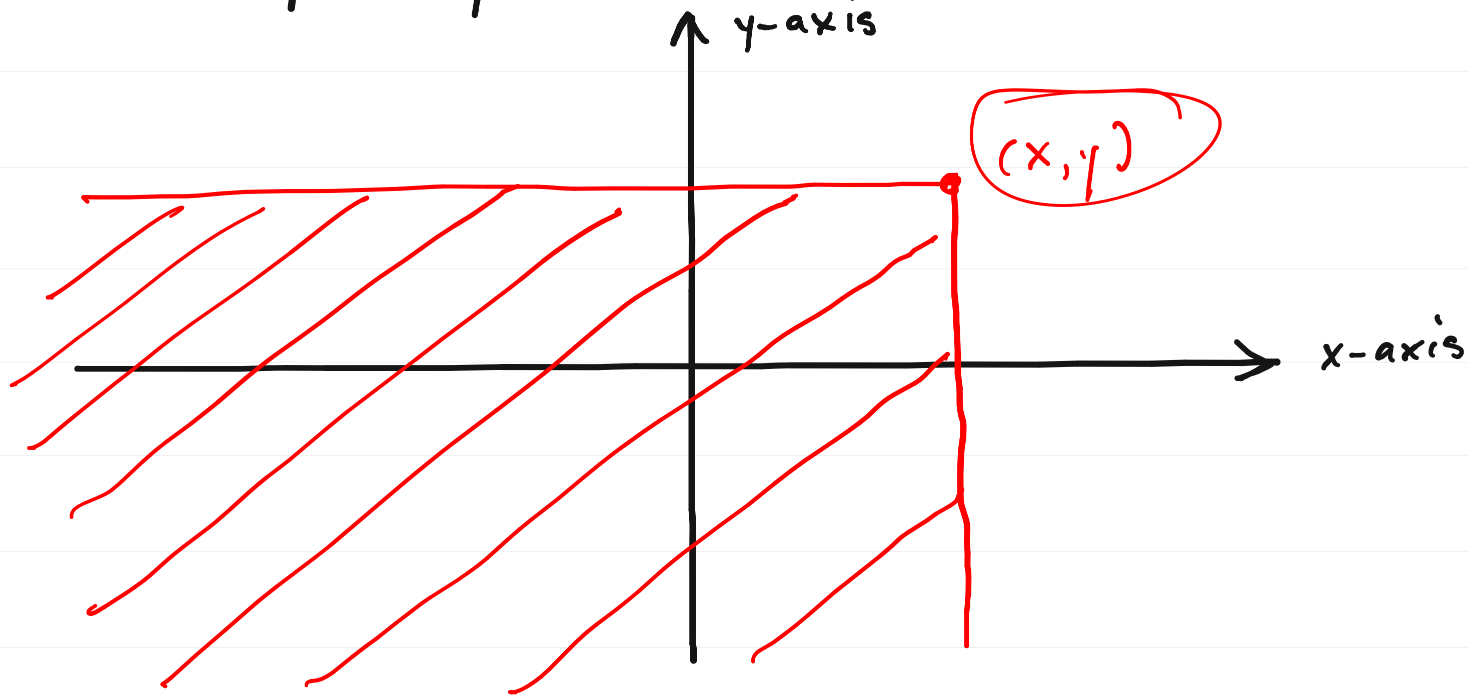
Defn: The joint cdf of two RVs defined on $(\mathcal{S}, \mathcal{F}, P)$ is the probability of the event

$$\{ \mathbb{X} \leq x \} \cap \{ \mathbb{Y} \leq y \} :$$

$$F_{\mathbb{X}\mathbb{Y}}(x,y) \triangleq P(\{ \mathbb{X} \leq x \} \cap \{ \mathbb{Y} \leq y \})$$
$$= P(\{ \omega \in \mathcal{S} : \mathbb{X}(\omega) \leq x \} \cap \{ \omega \in \mathcal{S} : \mathbb{Y}(\omega) \leq y \})$$

We specify $F_{\mathbb{X}\mathbb{Y}}(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

We can think of $F_{X,Y}(x,y)$ as the probability that (X,Y) falls within $D_1(x,y)$:



$$D_1(x,y) = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha \le x \text{ and } \beta \le y\}.$$

We will use the shorthand notation

$$\text{"}\{X \le x, Y \le y\}\text{"} = \{X \le x\} \cap \{Y \le y\}.$$

Properties of the Joint CDF:

1. $F_{XY}(-\infty, y) = 0$ and $F_{XY}(x, -\infty) = 0$

$F_{XY}(+\infty, y) = F_Y(y)$ and $F_{XY}(x, +\infty) = F_X(x)$

$F_{XY}(+\infty, +\infty) = 1$

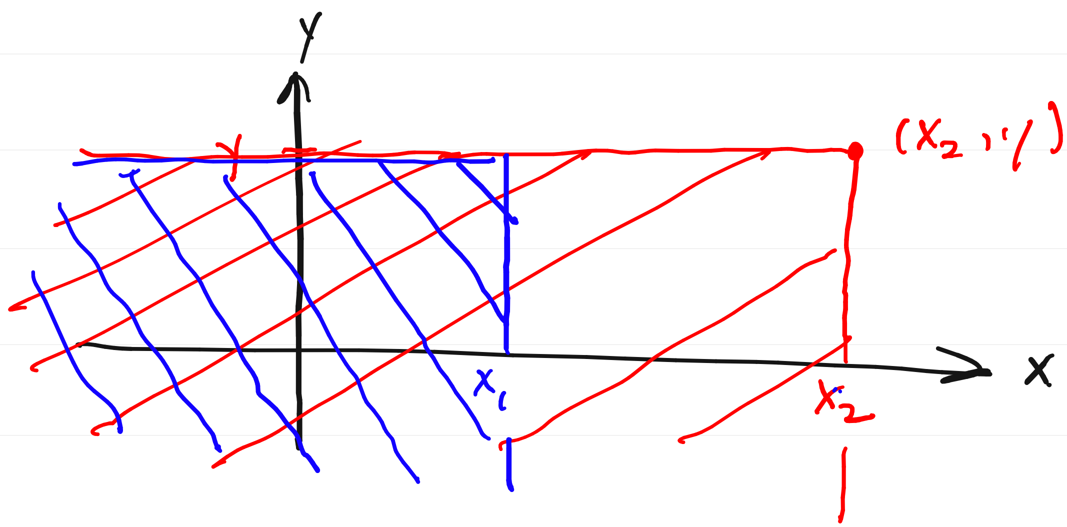
$\therefore$ If I know $F_{XY}(x,y)$, I can find $F_X(x)$ and $F_Y(y)$.

2. $P(\{x_1 < X \le x_2\} \cap \{Y \le y\})$

$$= F_{XY}(x_2, y) - F_{XY}(x_1, y)$$

and $P(\{X \le x\} \cap \{y_1 < Y \le y_2\})$

$$= F_{XY}(x, y_2) - F_{XY}(x, y_1).$$

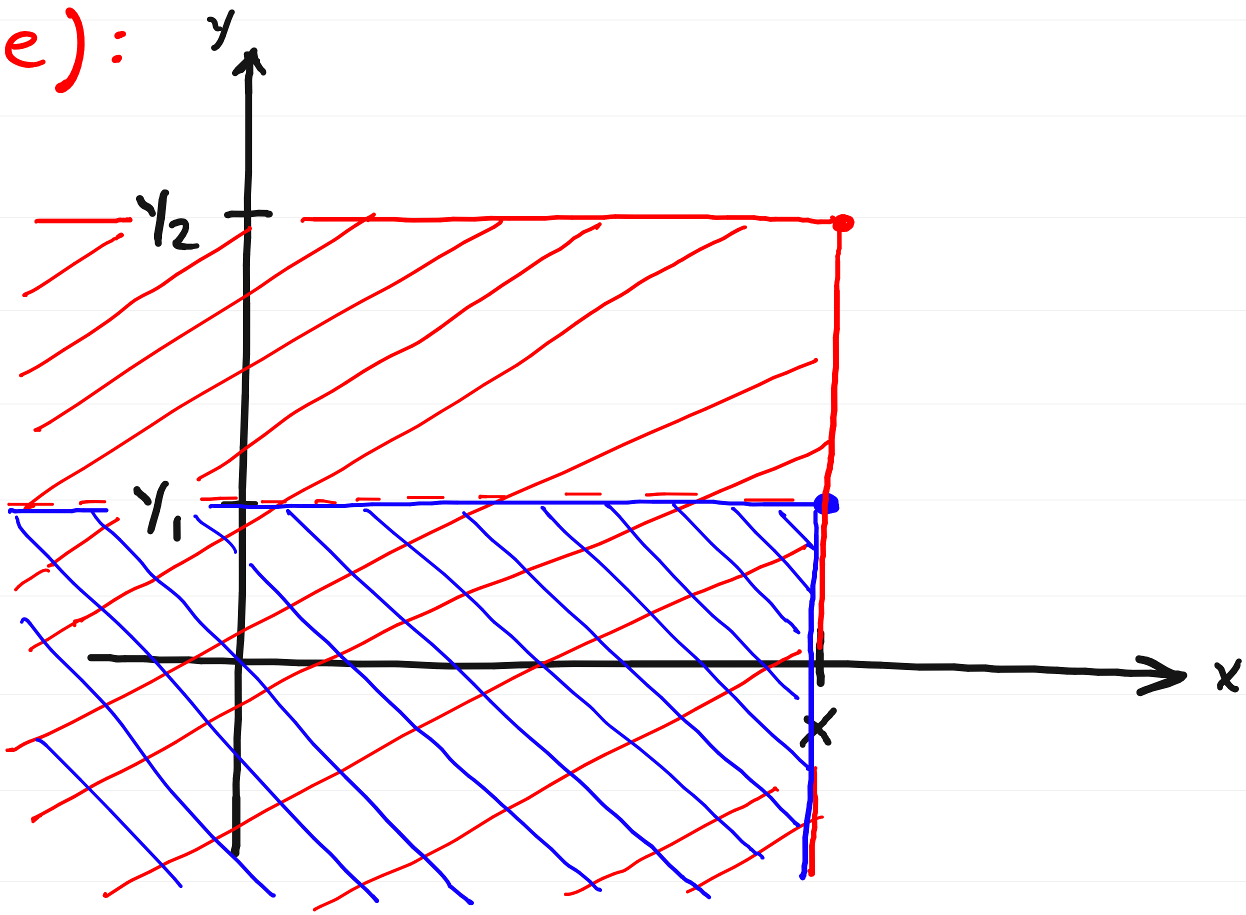


Similarly

$$P(\{X \le x\} \cap \{y_1 < Y \le y_2\})$$

$$= \underline{F_{XY}(x, y_2)} - \underline{F_{XY}(x, y_1)}$$

(exercise):



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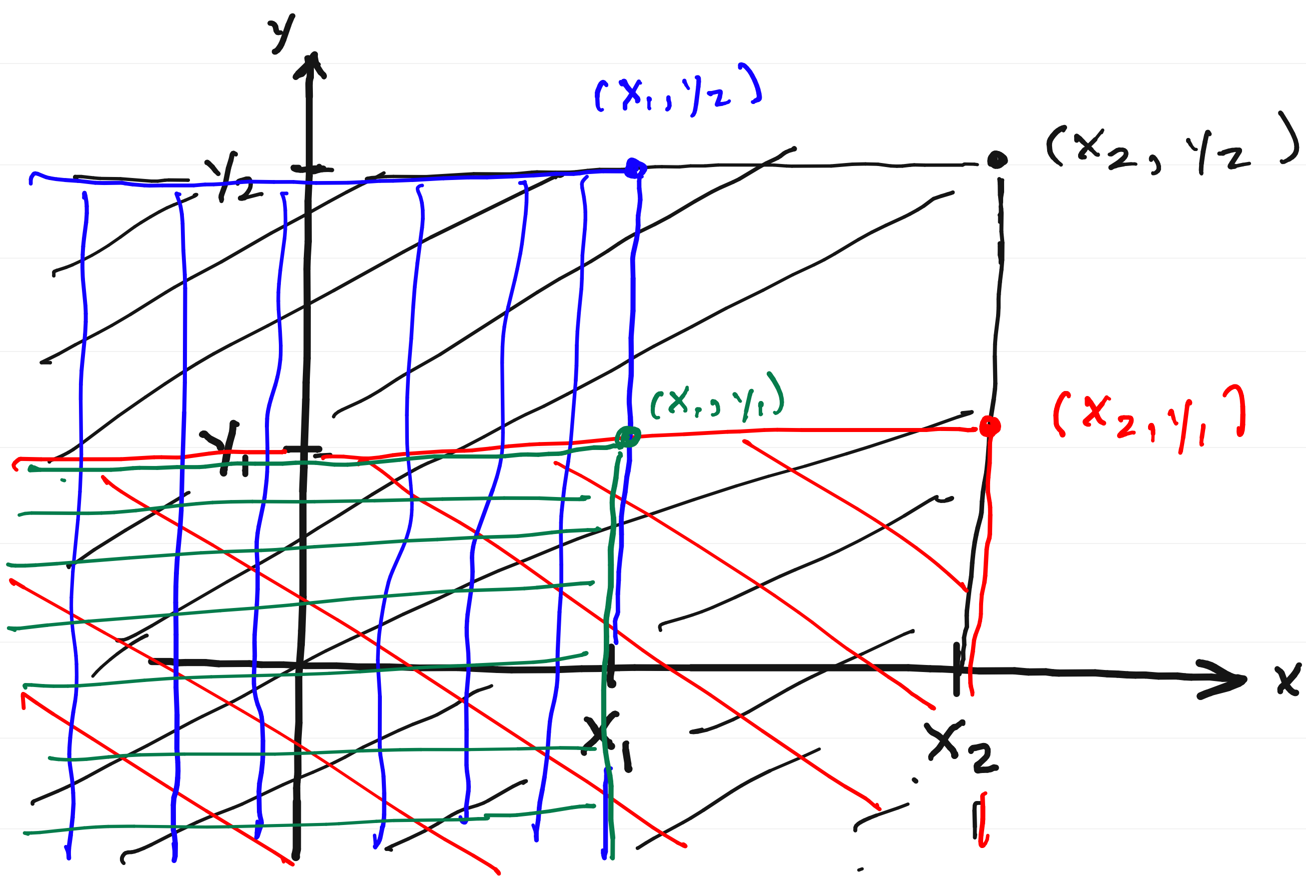
3. $P(\{x_1 < X \le x_2\} \cap \{y_1 < Y \le y_2\})$

$$= F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1).$$



Defn: The joint pdf of two RVs X and Y defined on $(S, \mathcal{F}, P)$ and having joint cdf $F_{XY}(x,y)$ is

$$f_{XY}(x,y) \triangleq \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}$$

Properties of the joint pdf:

(i) $f_{XY}(x,y) \geq 0$

(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

(iii) $\displaystyle\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(\alpha,\beta)\, d\alpha\, d\beta = F_{XY}(x,y)$

(iv) For any $D \in \mathcal{B}(\mathbb{R}^2)$

$$P(\{(X,Y) \in D\}) = \iint_D f_{XY}(x,y)\,dx\,dy$$

$$= \iint_{\mathbb{R}^2} f_{XY}(x,y) \cdot \mathbb{1}_D((x,y))\,dx\,dy$$

Two RVs defined on the same random experiment $(\mathcal{S}, \mathcal{F}, P)$ are called jointly distributed.

They will have a joint cdf and a joint pdf if continuous or a joint pmf if discrete.

If $f_{XY}(x,y)$ is the joint pdf (j-pdf) of two j-dist RVs X and Y,

Then

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

These two pdfs are called <u>marginal pdfs</u> of X and Y.

Defn: Two jointly distributed RVs X and Y are jointly Gaussian if their joint pdf (j-pdf) is of the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2r\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}, \quad \text{--- } (*)$$

where $\mu_X, \mu_Y \in \mathbb{R}$,

$\sigma_X, \sigma_Y > 0$,

and

$-1 \le r \le 1$. $(-1 < r < 1$ for pdf to exist.$)$

n.b. If X and Y are j-Gaussian,

then

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\}.$$

The converse is not true. (see Papoulis for eg.)