Session 2

Recall . . .

There are 3 fundamental set operations we have just defined:

Union: $A \cup B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ or } \omega \in B\}$

Intersection: $A \cap B \triangleq \{\omega \in \mathcal{S} : \omega \in A \text{ and } \omega \in B\}$

Complement: $\bar{A} \triangleq \{\omega \in \mathcal{S} : \omega \notin A\}$

These are the three fundamental set operations, but there are two other "set difference operations" that are sometimes used:

Recall . . .

2.2

Defn: An <u>indexed collection of sets</u> is a set of sets

$$\{A_i \ , i \in I\},$$

where I is an index set.

- So $\{A_i \ ; i \in I\}$ is a "set of sets" or a "family of sets" or a "collection of sets."

Recall . . .

Some Typical index Sets I:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ = natural numbers.

$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ = non-negative integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ = integers

$I_n = \{0, 1, 2, \ldots, n-1\}$

$\mathbb{R} = (-\infty, +\infty)$ = real line

We are interested in the "size" or cardinality of sets:

Defn: A set is finite if it has a finite number of elements.
(i.e., its elements can be put in one-to-one correspondence with the numbers $1, 2, \dots, n$ for some natural number n.

Defn: A set is infinite if it is not finite.

Infinite sets come in two varieties:

countable and uncountable.

Defn: An infinite set is countable if its elements can be put in one-to-one correspondence with the natural (counting) numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Defn: An infinite set is uncountable if it is not countable.

<u>e.g.</u> The following are examples of uncountable sets:

- $\mathbb{R} = (-\infty, +\infty)$
- $[0,1]$ and $(0,1)$
- $[a,b]$, $[a,b)$, $(a,b]$, (a,b)

  $\forall a, b \in \mathbb{R}$ such that $a < b$.

We now expand the definitions of union and intersection beyond simple binary operations:

Defn: Given an indexed family of sets

$$\{A_i ; i \in I\},$$

the <u>union</u> of the sets in the family is

$$\bigcup_{i \in I} A_i \triangleq \{\omega \in \mathcal{S} : \omega \in A_i \text{ for at least one } i \in I\}$$

the <u>intersection</u> of the sets in the family is

$$\bigcap_{i \in I} A_i \triangleq \{\omega \in \mathcal{S} : \omega \in A_i \text{ for all } i \in I\}$$

<u>Defn</u>: If $G \subset \mathcal{S}$ and $\{A_i \,;\, i \in I\}$ is a family of sets, then if

$$\bigcup_{i \in I} A_i = G$$

we say that $\{A_i \,;\, i \in I\}$ is <u>collectively exhaustive</u> over G.

<u>Defn</u>: A family of sets $\{A_i \,;\, i \in I\}$ is <u>disjoint</u> if

$$A_i \cap A_j = \phi, \quad \forall\, i, j \in I$$

such that $i \neq j$.

Defn: If $G \subset \mathcal{S}$ and

$\{A_i ; i \in I\}$ is a family sets,

then if

$$\bigcup_{i \in I} A_i = G$$

we say that $\{A_i ; i \in I\}$ is

collectively exhaustive over G

Defn: A family of sets $\{A_i ; i \in I\}$

is disjoint, if

$$A_i \cap A_j = \phi, \quad \forall i, j \in I$$

such that $i \neq j$.

Defn: A family of sets $\{A_i ; i \in I\}$ is a partition of $\mathcal{S}$ if it is disjoint and collectively exhaustive over $\mathcal{S}$.

n.b. $\{A_i ; \in I\}$ is a partition of $G \subset \mathcal{S}$ if it is disjoint and

$$\bigcup_{i \in I} A_i = G.$$

Fact: Let $\{A_i \,;\, i \in I\}$ be a partition of $\mathcal{S}$.

Define

$$B_i \triangleq A_i \cap G \,, \quad i \in I$$

where $G \subset \mathcal{S}$.

Then $\{B_i \,;\, i \in I\}$ is a partition of G.

Proof: Homework

Probability Spaces

A probability space $(\mathcal{S}, \mathcal{F}, P)$ is a triple made up of 3 elements:

1. Sample space $\mathcal{S}$.

2. A collection of events (subsets of $\mathcal{S}$) $\mathcal{F}(\mathcal{S})$

3. The probabilities $P(A)$ for each event in the event space

$$P : \mathcal{F}(\mathcal{S}) \longrightarrow [0,1]$$

The Sample Space $\mathcal{S}$

Defn: The sample space $\mathcal{S}$ is a non-empty set of possible outcomes of a random experiment.

$\boxed{\text{One and only one outcome from the sample space occurs when we perform a random experiment.}}$

The Event Space $\mathcal{F}(\mathcal{S})$

Defn: The <u>event space</u> $\mathcal{F}(\mathcal{S})$ is a non-empty collection of subsets of $\mathcal{S}$ satisfying the following closure properties.

1. If $A \in \mathcal{F}(\mathcal{S})$, then $\bar{A} \in \mathcal{F}(\mathcal{S})$.
2. For any finite n, if $A_i \in \mathcal{F}(\mathcal{S})$ for $i = 1, 2, \ldots, n$, then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}.$$

3. If $A_i \in \mathcal{F}(\Omega)$, $i = 1, 2, 3, \ldots$ then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\Omega).$$

A sets of subsets satisfying these 3 properties is called a <u>σ-field</u>.

(n.b. If only 1 and 2 hold, you have a <u>field</u> of sets.)

What about intersections?

Proof: Let $A, B \in \mathcal{F}$. Then

$$A \cap B = \overline{\overline{(A \cap B)}}$$
$$= \overline{\bar{A} \cup \bar{B}} \in \mathcal{F} \quad (\text{De Morgan})$$

Since
$$\bar{A} \in \mathcal{F}(\mathcal{S}) \quad (\text{closure Prop. 1})$$
$$\bar{B} \in \mathcal{F}(\mathcal{S}) \quad (\text{closure Prop. 1}$$
$$\bar{A} \cup \bar{B} \in \mathcal{F}(\mathcal{S}) \quad (\text{closure Prop. 2})$$
$$\overline{\bar{A} \cup \bar{B}} \in \mathcal{F}(\mathcal{S}) \quad (\text{closure Prop. 1})$$

But $\overline{\bar{A} \cup \bar{B}} = A \cup B$ ■

It follows from the closure properties that $\phi, \mathcal{S} \in \mathcal{F}(\mathcal{S})$.

Proof: Suppose $A \in \mathcal{F}(\mathcal{S})$ ($\mathcal{F}(\mathcal{S})$ is non-empty)

then $\bar{A} \in \mathcal{F}(\mathcal{S})$ (prop. 1)

Furthermore

$$\mathcal{S} = A \cup \bar{A} \in \mathcal{F}(\mathcal{S}) \quad \text{(prop. 2)}$$

$$\bar{\mathcal{S}} = \phi \in \mathcal{F}(\mathcal{S}) \quad \text{(prop. 1)}$$

$$\therefore \phi, \mathcal{S} \in \mathcal{F}(\mathcal{S}).$$

Probability Measure

Defn: A probability measure $P(\cdot)$ (corresponding to $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$) is an assingment of a real number $P(A)$ to each $A \in \mathcal{F}(\mathcal{S})$ satisfying the Axioms of Probability

Axioms of Probability

1. $P(A) \geq 0, \quad \forall A \in \mathcal{F}(\mathcal{S})$.

2. $P(\mathcal{S}) = 1$.

3. If $A_1, A_2 \in \mathcal{F}(\mathcal{S})$ and $A_1 \cap A_2 = \phi$,
   then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

- If $\{A_1, \ldots, A_n\}$ (finite) are <u>disjoint</u> $(A_j \cap A_k = \phi,\ j \neq k)$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

4. If $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F}(\mathcal{S})$ is a countable collection of disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

<u>n.b.</u> $P(\cdot)$ is a set function.

$P(\cdot) : \mathcal{F}(\mathcal{S}) \to \mathbb{R}.$