Session 21

Recall...

Theorem: Let X and Y be two j-dist, independent RVs with marginal pdfs $f_X(x)$ and $f_Y(y)$, respectively. Then the pdf of their sum $Z = X + Y$ is given by the convolution

$$f_Z(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(z-y)\, dy.$$

Example: Let $\mathbb{X}$ and $\mathbb{Y}$ be two i-dist independent exponential RVs, both with mean μ. Let

$$\mathbb{Z} = \mathbb{X} + \mathbb{Y}.$$

Find $f_{\mathbb{Z}}(z)$.

$$f_{\mathbb{Z}}(z) = \int_{-\infty}^{\infty} f_{\mathbb{Y}}(y)\, f_{\mathbb{X}}(z-y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\mu} \exp\left(\frac{-y}{\mu}\right) \cdot 1_{[0,\infty)}(y) \cdot \frac{1}{\mu} \exp\left(-\frac{(z-y)}{\mu}\right) 1_{[0,\infty)}(z-y)\, dy$$

$$= \int_{0}^{z} \frac{1}{\mu^2} \exp\left(\frac{-z}{\mu}\right) dy = \frac{z}{\mu^2} \exp\left(\frac{-z}{\mu}\right) \cdot 1_{[0,\infty)}(z).$$

Two Functions of Two RVs

Given two RVs X and Y with j-pdf $f_{XY}(x,y)$, and given two new RVs

$$Z = g(X,Y),$$

$$W = h(X,Y),$$

we want to find $f_{ZW}(z,w)$.

We will start by finding $F_{ZW}(z,w)$, the joint cdf.

$$F_{ZW}(z,w) = P(\{Z \le z\} \cap \{W \le w\})$$
$$= P(\{(X,Y) \in D_{zw}\})$$

where $D_{zw} \triangleq \{(x,y) \in \mathbb{R}^2 : g(x,y) \le z \text{ and } h(x,y) \le w\}$

$$\therefore \quad F_{ZW}(z,w) = \iint_{D_{zw}} f_{XY}(x,y)\, dx\, dy \quad , \forall z,w \in \mathbb{R}$$

and then

$$f_{ZW}(z,w) = \frac{\partial^2 F_{ZW}(z,w)}{\partial z\, \partial w} .$$

Direct Joint Density Determination

Theorem: Let X and Y be two j-dist RVs with j-pdf $f_{XY}(x,y)$. Let $Z = g(X,Y)$ and $W = h(X,Y)$, and assume the functions $g(x,y)$ and $h(x,y)$ satisfy the following conditions:

(1) The equations $z = g(x,y)$ and $w = h(x,y)$ can be uniquely (simultaneously) solved for x and y in terms of z and w.

(2) The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous.

...

(Theorem continued)

Then the j-pdf of Z and W is

$$f_{ZW}(z,w) = f_{XY}(x(z,w), y(z,w)) \left| \frac{\partial(x,y)}{\partial(z,w)} \right| ,$$

where the Jacobian is the determinant

$$\frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \frac{\partial x}{\partial z} \cdot \frac{\partial y}{\partial w} - \frac{\partial y}{\partial z} \cdot \frac{\partial x}{\partial w} .$$

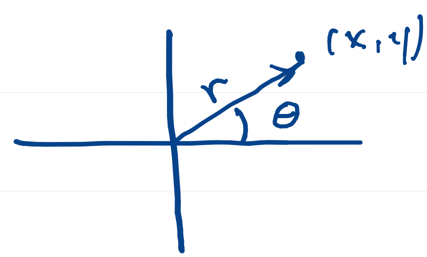
<u>Proof</u>: See Papoulis

Example: Let X and Y be two zero-mean i.i.d (independent, identically distributed) Gaussian RVs, both with variance σ^2:

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{(x^2+y^2)}{2\sigma^2} \right\}.$$

Let $R \triangleq \sqrt{X^2+Y^2}$ and $\Theta \triangleq \tan^{-1}(Y,X)$

Find $f_{R\Theta}(r,\theta)$.



$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y,x)$

$x(r,\theta) = r\cos\theta$

$y(r,\theta) = r\sin\theta$

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$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$= r$$

$$\therefore f_{R\Theta}(r,\theta) = f_{X,Y}(x(r,\theta), y(r,\theta)) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$$

$$= \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{(r^2\cos^2\theta + r^2\sin^2\theta)}{2\sigma^2} \right\} \cdot |r|$$

$$= \frac{r}{2\pi\sigma^2} \exp\left\{ \frac{-r^2}{2\sigma^2} \right\} \cdot 1_{[0,\infty)}(r) \cdot 1_{(-\pi,\pi]}(\theta)$$

n.b.

$$f_{R}(r) = \int_{-\infty}^{\infty} f_{R\Theta}(r,\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2}\right) \cdot 1_{[0,\infty)}(r) \quad d\theta$$

$$= \frac{r}{\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2}\right) \cdot 1_{[0,\infty)}(r) \; .$$

$$f_{\Theta}(\theta) = ?$$ (exercise)

We can loosen the constraints (1) and (2) of the last theorem...

<u>Theorem:</u> Let $\mathbb{X}$ and $\mathbb{Y}$ be two j-dist RVs with j-pdf $f_{\mathbb{X}\mathbb{Y}}(x,y)$, and let

$$\mathbb{Z} = g(\mathbb{X},\mathbb{Y}) \quad \text{and} \quad \mathbb{W} = h(\mathbb{X},\mathbb{Y}).$$

To find $f_{\mathbb{Z}\mathbb{W}}(z,w)$, we must find all real solutions (x_n, y_n) such that

$$g(x_n,y_n) = z \quad \text{and} \quad h(x_n,y_n) = w,$$

$$\text{for } n = 1, 2, \ldots, N.$$

then

$$f_{ZW}(z,w) = f_{XY}(x_1(z,w), y_1(z,w)) \left| \frac{\partial(x_1, y_1)}{\partial(z,w)} \right|$$

$$+ \cdots + f_{XY}(x_N(z,w), y_N(z,w)) \left| \frac{\partial(x_N, y_N)}{\partial(z,w)} \right|$$

$$= \sum_{n=1}^{N} f_{XY}(x_n(z,w), y_n(z,w)) \cdot \left| \frac{\partial(x_n, y_n)}{\partial(z,w)} \right|.$$

Auxiliary Variables

- Sometimes, we want to find the pdf of

$$\mathbb{Z} = g(\mathbb{X}, \mathbb{Y})$$

when $f_{\mathbb{X}\mathbb{Y}}(x, y)$ is given.

- It's often easier to use the direct technique $(\mathbb{X}, \mathbb{Y}) \longmapsto (\mathbb{Z}, \mathbb{W})$ to find $f_{\mathbb{Z}}(z)$.

- But we only have one RV $\mathbb{Z}$.
  What do we do?

- You introduce an (arbitrary) <u>auxiliary RV</u>:

$$\mathbb{W} = h(\mathbb{X}, \mathbb{Y}).$$

- Then you find $f_{\mathbb{Z}\mathbb{W}}(z,w)$ using the direct pdf method.

- Then you integrate over w to get $f_{\mathbb{Z}}(z)$:

$$f_{\mathbb{Z}}(z) = \int_{-\infty}^{\infty} f_{\mathbb{Z}\mathbb{W}}(z,w)\, dw.$$

How do you pick the aux. RV $\mathbb{W} = h(\mathbb{X}, \mathbb{Y})$?

Popular choices include

$$\mathbb{W} = \mathbb{X} \quad \text{or} \quad \mathbb{W} = \mathbb{Y}$$

or

$$\text{If} \quad \mathbb{Z} = \sqrt{\mathbb{X}^2 + \mathbb{Y}^2}$$

$$\text{or} \quad \mathbb{Z} = g(\mathbb{X}^2 + \mathbb{Y}^2)$$

You can pick

$$\mathbb{W} = \tan^{-1}(y, x)$$

In general, it can be trial-on-error guessing.

Joint Moments

Given two RVs X and Y and a RV

$$Z = g(X, Y),$$

we know that

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz.$$

It is often easier to express $E[Z]$ in terms of $f_{XY}(x,y)$ and $g(x,y)$.

<u>Theorem:</u> Given two j-dist RVs X and Y with j-pdf $f_{XY}(x,y)$, and a new RV $Z = g(X, Y)$, we can compute

$$E[Z] = E[g(X,Y)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

Proof: (Outline)

Let $\Delta D_z \triangleq \{(x,y) \in \mathbb{R}^2 : z < g(x,y) \leq z + \Delta z\}$

Then for each differential element in the Riemann integral

$$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz$$

there is a corresponding ΔD_z in the xy-plane

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As dz covers the whole real line, the corresponding ΔD_z covers the whole x-y plane in $\mathbb{R}^2$.
These ΔD_z do not overlap.

$$\int_{-\infty}^{\infty} z\, f_Z(z)\, dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$\therefore$ We can compute $E[g(X,Y)]$ as

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy$$

$\blacksquare$

Linearity of Expectation

Let X and Y be two j-dist RVs with j-pdf $f_{XY}(x,y)$, and let

$$g_1(X,Y),\ g_2(X,Y),\ \ldots,\ g_N(X,Y)$$

be functions of X and Y.

Then for constants $\alpha_1, \alpha_2, \ldots, \alpha_N$,

$$E\left[\sum_{n=1}^{N} \alpha_n g_n(X,Y)\right] = \sum_{n=1}^{N} \alpha_n E\left[g_n(X,Y)\right].$$

Proof: Exercise

Defn: Given two j-dist RVs X and Y, the correlation between X and Y is defined as

$$\mathrm{corr}(X, Y) \triangleq E[XY];$$

the covariance between X and Y is defined as

$$\mathrm{cov}(X, Y) \triangleq E[(X-\bar{X})(Y-\bar{Y})];$$

The correlation coefficient between X and Y is defined as

$$r_{XY} \triangleq \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$

<u>n.b.</u> $-1 \leq r_{XY} \leq 1$

<u>Fact</u>: If X and Y are statistically independent, then $r_{XY} = 0$.

<u>Proof</u>: (Exercise).

But, the converse is not true!

There is a special case where

$$r_{XY} = 0 \implies X \perp\!\!\!\perp Y$$

when X and Y are jointly Gaussian.

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In this case $r_{XY} = r$ in the j-Gaussian pdf.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2r\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}\Bigg|_{r=0}$$

$$\frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

$$= f_X(x) \cdot f_Y(y)$$

$$\Rightarrow \quad X \perp\!\!\!\perp Y \checkmark$$

<u>Defn:</u> Two RVs X and Y are <u>uncorrelated</u> if their covariance is equal to zero.

This is true if any one of the following equivalent conditions is true:

1. $\text{Cov}(X, Y) = 0$

2. $r_{XY} = 0$

3. $E[XY] = E[X] \cdot E[Y]$.

<u>Defn:</u> Two RVs X and Y are <u>orthogonal</u> if $E[XY] = 0$.

An example of a situation where X and Y are uncorrelated but not independent:

Suppose $X \sim N[0, \sigma_x^2]$

Define $Y = |X|$ (Y is <u>not</u> normal.)
(exercise)

$$E[XY] = E[X \cdot |X|] = 0 \Rightarrow r_{xy} = 0$$

$$\text{cov}(X, Y) = \cancelto{0}{E[XY]} - \cancelto{0}{E[X]} \cdot E[Y] = 0$$

$\therefore$ X and Y are uncorrelated

But clearly Y and X are not independent.
($Y = |X|$)

Fact: If $E[X^2] < \infty$ and $E[Y^2] < \infty$,

then

$$|E[XY]| \leq \sqrt{E[X^2] \cdot E[Y^2]},$$

with equality iff

$$Y = a_0 X, \qquad \text{(a.e.)}$$

almost everywhere.

for some constant a_0.

Proof: $E[(aX - Y)^2] \geq 0$