Session 26

Convergence of Sequences of RVs

$$\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n, \dots$$

Example: Suppose I make a sequence of measurements:

$$\mathbb{X}_k = a + \mathbb{W}_k, \quad k = 1, 2, 3, \dots$$

a = parameter of interest.

$\mathbb{X}_k$ = k-th measurement.

$\mathbb{W}_k$ = experimental error in the k-th measurement

$E[\mathbb{W}_k] = 0$ (unbiased measurement)

If we consider $X_1, \dots, X_n \dots$ to be measurements, we typically estimate the value of a by

$$Y_n = \frac{1}{n}\left[X_1 + X_2 + \cdots + X_n\right]$$

Is this a good estimate of a?

Hopefully, $Y_n \longrightarrow a$, as $n \to \infty$.

Is this true? Is it always true?

Is it ever true?

What does $Y_n \longrightarrow a$ mean?

Note that

$$\mathbb{Y}_n = \frac{1}{n}\left[\mathbb{X}_1 + \cdots + \mathbb{X}_n\right]$$

is itself a random variable, so

$$\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \ldots, \mathbb{Y}_n, \ldots$$

is also a sequence of random variables
(i.e., a random sequence.)

<u>Q</u>: What does it mean to ask if

$$\mathbb{Y}_n \longrightarrow a \quad \text{as } n \longrightarrow \infty \text{ ?}$$

<u>Defn</u>: A <u>random sequence</u> or a discrete-time stochastic process is a sequence of random variables $X_1, \ldots, X_n, \ldots$ defined on $(\mathcal{S}, \mathcal{F}, P)$.

(If X_i is a R.V. for each $i \in \mathbb{N}$, then measurability is not an issue.)

- We often write a random sequence as $\{X_n\}$ or $\{X_n\}_{n \in \mathbb{N}}$ or $\{X_n\}_{n \geq 1}$
- For any specific $\omega_0 \in \mathcal{S}$ of $(\mathcal{S}, \mathcal{F}, P)$
  $X_1(\omega_0), X_2(\omega_0), \ldots, X_n(\omega_0), \ldots$
  is a sequence of real numbers.

Defn: A sequence of real numbers $x_1, \dots, x_n, \dots$ is said to <u>converge to a limit x</u> if, for $\forall \varepsilon > 0$, there exists a number $n_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \ , \ \forall n \geq n_\varepsilon .$$

" $x_n \to x$ as $n \to \infty$ "

Given a random sequence

$$\mathbb{X}_1(\cdot), \mathbb{X}_2(\cdot), \ldots, \mathbb{X}_n(\cdot), \ldots$$

for any particular $\in \mathcal{S}$, we have

$$\mathbb{X}_1(\omega), \mathbb{X}_2(\omega), \ldots, \mathbb{X}_n(\omega), \ldots$$

is a sequence of real numbers

- It may converge to a number $\mathbb{X}(\omega)$
- or, it may not converge.

n.b. The $\mathbb{X}(\omega)$ that the random sequence converges to is itself a function of ω ($\mathbb{X}(\omega)$ is a R.V.).

Given a random sequence

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for any particular $\omega \in S$, we have

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n.b. The $X(\omega)$ that the random sequence converges to is itself a function of ω ($X(\omega)$ is a R.V.).

Given a random sequence

$$\mathbb{X}_1(\omega), \dots, \mathbb{X}_n(\omega), \dots$$

most likely it will converge for some $\omega \in \mathcal{S}$, and not converge for other $\omega \in \mathcal{S}$.

When we study convergence of random sequences (Stochastic Convergence) we study the set $A \subset \mathcal{S}$ for which

$$\mathbb{X}_1(\omega), \dots, \mathbb{X}_n(\omega), \dots$$

is a convergent sequence for all $\omega \in A$.

Defn: We say that a sequence of RVs <u>converges everywhere</u> (e) if the sequences

$$\mathbb{X}_1(\omega), \mathbb{X}_2(\omega), \ldots, \mathbb{X}_n(\omega), \ldots$$

each converge to a number $\mathbb{X}(\omega)$ for each $\omega \in \Omega$.

n.b.

- The number $\mathbb{X}(\omega)$ that each sequence converges to for each $\omega \in \Omega$ is a function of ω. $\mathbb{X}(\omega)$ is a RV.
- Convergence everywhere is too strong or restrictive to be useful.

Defn: A random sequence $\{X_n\}$ <u>converges almost everywhere</u> (a.e.) if the set of outcomes $A \subset S$ such that $X_n(\omega) \xrightarrow{n \to \infty} X(\omega)$, $\omega \in A$ exists and has proba ility 1:

$$P(A) = 1.$$

Other names for convergence almost everywhere.

- also sure convergence (a.s.)
- convergence with probability one.

We write this as

$$\text{“} X_n \xrightarrow{a.e} X \text{”}$$

$$\text{“} P(\{X_n \to X\}) = 1 \text{”}$$

Defn: A random sequence $\{X_n\}$ converges in mean-square (m.s) to a RV X if

$$E[|X_n - X|^2] \to 0 \text{ as } n \to \infty.$$

n.b. If we have the j-pdf of X_n and X, we can compute

$$E[|X_n - X|^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_n - x)^2 f_{X_n X}(x_n, x)\, dx_n\, dx$$

So in principle, it is easy to determine mean-square convergence

<u>Defn</u>: A random sequence $\{X_n\}$ <u>converges in probability</u> (p) to a random variable X if, $\forall \varepsilon > 0$

$$P(\{|X_n - X| > \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

Defn: A random sequence $\{X_n\}$ <u>converges in distribution</u> (d) to a RV X if

$$F_{X_n}(x) \longrightarrow F_X(x) \quad \text{as } n \to \infty \quad (*)$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

(*): <u>i.e.</u> $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|F_{X_n}(x) - F_X(x)| < \varepsilon, \quad \forall n \geq n_\varepsilon$$

for all $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

<u>Defn</u>: A random sequence $\{X_n\}$ <u>converges in density</u> (den) to a R.V. X if

$$f_{X_n}(x) \longrightarrow f_X(x) \text{ as } n \to \infty$$

at every point $x \in \mathbb{R}$ where $F_X(x)$ is continuous.

Aren't convergence in density and convergence in distribution equivalent?

No!

Example: Let $\{X_n\}$ be a sequence of RVs having p.d.fs

$$f_{X_n}(x) = [1 + \cos(2\pi n x)] \cdot 1_{[0,1]}(x).$$

n.b. (i) $f_{X_n}(x) \geq 0, \ \forall x$

(ii) $= \int_{-\infty}^{\infty} f_n(x)\,dx = \int_0^1 (1 + \cos(2\pi n x))\,dx = \left(x + \frac{1}{2\pi n}\sin(2\pi n x)\right)\Big|_0^1$
$= 1, \ n = 1, 2, 3, \ldots$

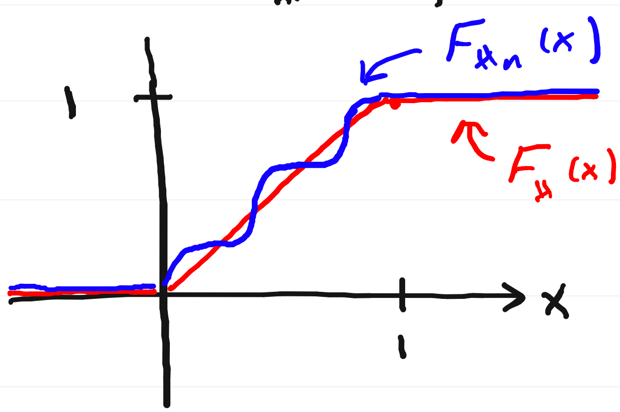
Now define

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & x \in [0,1], \\ 1, & x > 1. \end{cases}$$

Note that

$$F_{X_n}(x) = \int_{-\infty}^{x} f_n(\alpha)\, d\alpha = \begin{cases} 0, & x<0 \\ x + \frac{1}{2\pi n}\sin(2\pi n x), & x \in [0,1] \\ 1, & x > 1 \end{cases}$$

Clearly $F_{X_n}(x) \longrightarrow F_X(x)$, $\forall x \in \mathbb{R}$



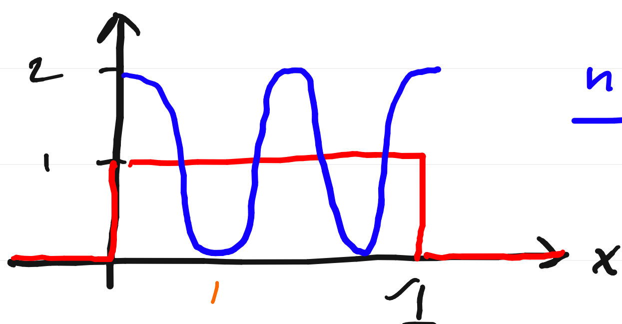
$F_{X_n}(x) \to F_X(x)$ as $n \to \infty$.

But $f_{X_n}(x) \not\to f_X(x)$

$$f_X(x) = \frac{d F_X(x)}{dx} = 1_{[0,1]}(x)$$

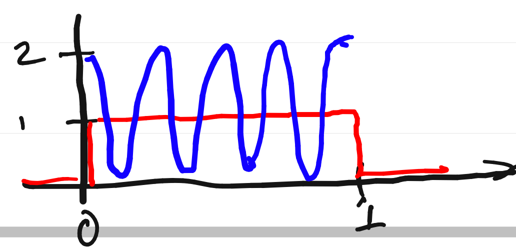
and

$$f_n(x) = [1 + \cos 2\pi n x] \cdot 1_{[0,1]}(x)$$



$n = 2$

As $n \to \infty$



$f_{X_n}(x) \not\to f_X(x)$ as $n \to \infty$.

$\therefore$ Convergence (d) $\not\Rightarrow$ convergence (den)

However

$$\text{convergence (den)} \Rightarrow \text{convergence (d)}.$$

(n.b. The integration that converts $f_{X_n}(x)$ into $F_{X_n}(x)$ smoothes things out.)

Cauchy Criterion for Convergence

26.19

Recall that a sequence of real numbers $x_1, \dots, x_n, \dots$ converges to a limit x if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon , \quad \forall n \geq n_\varepsilon .$$

To use this definition to determine if $x_1, \dots, x_n, \dots$ converges, we have to know the limit x.

The Cauchy Criterion gives us a way to test for convergence without knowing the limit x.

Cauchy Criterion: If $\{X_n\}$ is a sequence of real numbers and

$$|X_{n+m} - X_n| \to 0 \quad \text{as} \quad n \to \infty$$

for all $m \in \mathbb{N}$, then the sequence converges to a real number.

n.b. The Cauchy criterion can be applied to various forms of stochastic convergence.

e.g. $E\left[|X_{n+m} - X_n|^2\right] \to 0$ as $n \to \infty$ for all $m = 1, 2, 3, 4 \ldots$

then $\{X_n\}$ converges in mean-square.

Comparison of Modes of Convergence