Session 6

Recall...

Example of a set A where the Riemann integral does not exist

Suppose I have a pdf

$$f(r) = 1_{[0,1]}(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $A = \mathbb{Q}$ = rational numbers.

$$P(A) = P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r) \cdot 1_{\mathbb{Q}}(r)\, dr$$

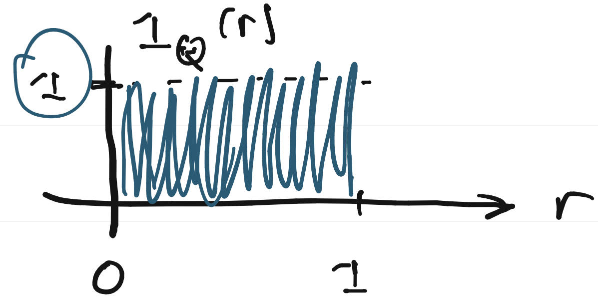
$$= \int_0^1 1_{\mathbb{Q}}(r)\, dr$$

does not exist as a Riemann integral

Recall...

6.2

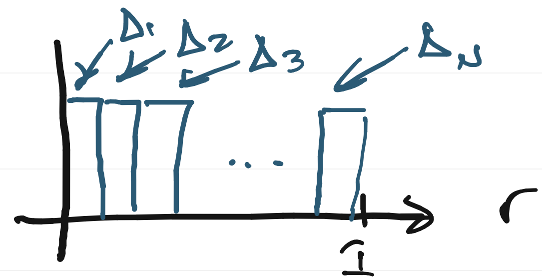
$$P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r)\cdot 1_{\mathbb{Q}}(r)\,dr$$



$$= \int_{0}^{1} 1_{\mathbb{Q}}(r)\,dr$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} |\Delta_k|\cdot 1_{\mathbb{Q}}(\hat{x}_k)$$

$$\hat{x}_k \in \Delta_k$$



$$\underline{\text{Sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\min_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 0 = 0$$

$$\overline{\text{Sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\max_{x\in\Delta_k} 1_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 1 = 1$$

For a Riemann integral to exist

$$\lim_{N\to\infty} \underline{\text{Sum}}_N = \lim_{N\to\infty} \overline{\text{Sum}}_N .$$

$\therefore$ The Riemann integral does not exist in this case.

But here, we have

$$\lim_{N\to\infty} \overline{\text{sum}}_N \neq \lim_{N\to\infty} \underline{\text{sum}}_N$$

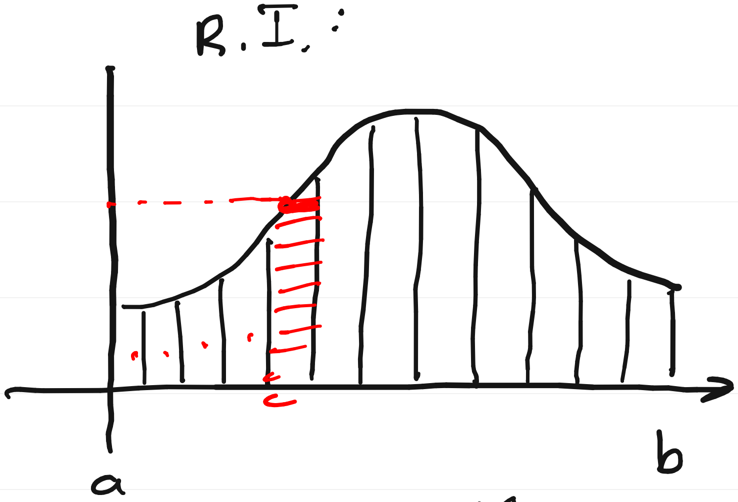
$\therefore$ R.I. doesn't exist

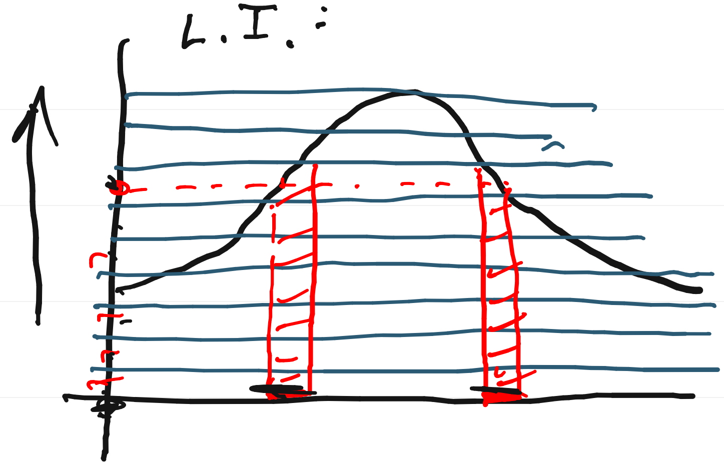
Yet by intuition

$$\underbrace{P(\mathbb{Q})} = \int_0^1 \mathbb{1}_{\mathbb{Q}}(r)\,dr = 0$$

We have a problem

This is why the Lebesgue was introduced.





1. $\int_{-\infty}^{\infty} 1_A(x)\, dx$ is defined for all $A \in B(\mathbb{R})$ if I use a Lebesgue integral

2. If the integrand has bounded absolute value, we can exchange the order of limits

How do we reconsile the difference between the R.I. and the L.I. ?

Important Fact: If the Rieman Integral exists, the Lebesgue integral exists, and they are equal.

Proof: Measure Theory Course (MA 544)

This leads to the "Engineering Compromise on integration"

Engineering Compromise on Integration

Given $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, for events $A \in \mathcal{B}(\mathbb{R})$ we calculate probabilities as

$$P(A) = \int_A f(r)\,dr = \int_{-\infty}^{\infty} f(r) \cdot \mathbb{1}_A(r)\,dr \quad (\text{L.I.})$$

we interpret this as a Lebesgue integral, however for "friendly functions" (i.e., Riemann integrable functions) we compute the value of $P(A)$ using Riemann integration.

Compromise Approach is not Uncommon

	Theory	Practice
Computation:	Real	finite subset of the rationals.
Integration:	Lebesgue	Riemann

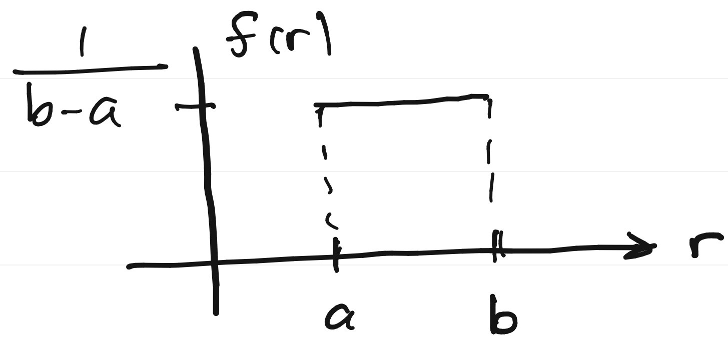
So we actually compute probabilities using the R.I. whenever possible.

We invoke the properties of the L.I. whenever it is beneficial to do so.

There are a number of useful pdfs we will want to know.

Ex. 1 The uniform pdf

$$f(r) = \frac{1}{b-a} \cdot \mathbb{1}_{[a,b]}(r), \quad r \in \mathbb{R}, \; a, b \in \mathbb{R}, \; a < b.$$



$$= \begin{cases} \frac{1}{b-a}, & r \in [a,b] \\ 0, & \text{elsewhere.} \end{cases}$$

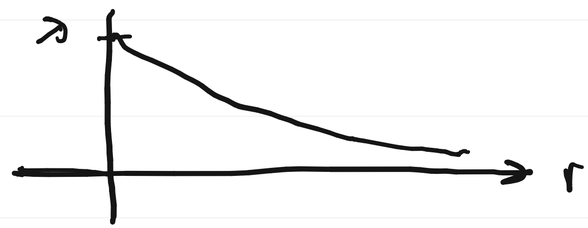
n.b. (i) $f(r) \geq 0, \; \forall r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\, dr = 1.$

Ex. 2 The exponential pdf

$$f(r) = \lambda e^{-\lambda r} \cdot 1_{[0,\infty)}(r) = \begin{cases} \lambda e^{-\lambda r}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

$\lambda > 0$

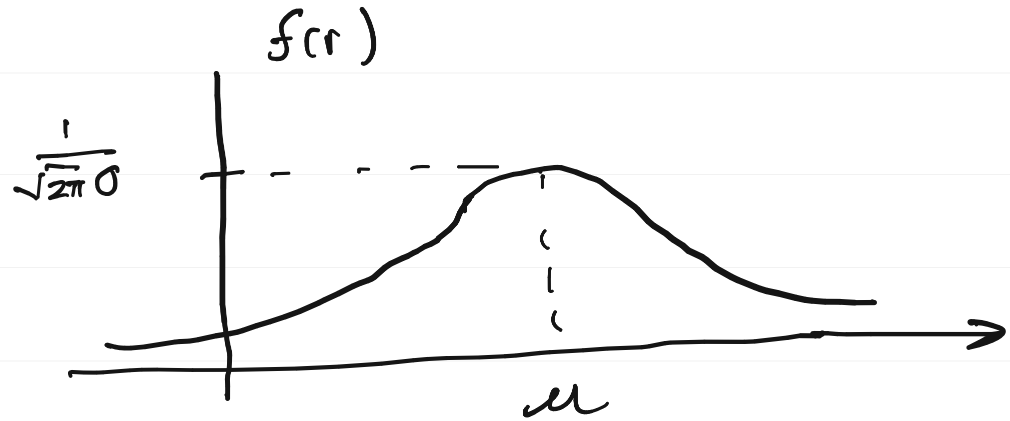


n.b. (i) $f(r) \geq 0, \quad \forall r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\, dr = 1.$

Ex. 3 The Gaussian pdf

$$f(r) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(r-\mu)^2}{2\sigma^2} \right\} \qquad \begin{array}{l} r \in \mathbb{R} \\ \mu \in \mathbb{R} \\ \sigma > 0 \end{array}$$



(i) $f(r) \geq 0 \quad , r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\,dr = 1$

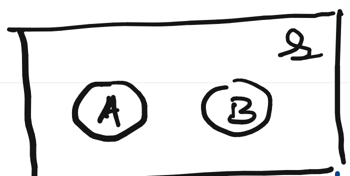
PMFs and PDFs You Should Know

PMFs	PDFs
Binomial	Uniform
Geometric	Exponential
Poisson	Gaussian

Conditional Probability

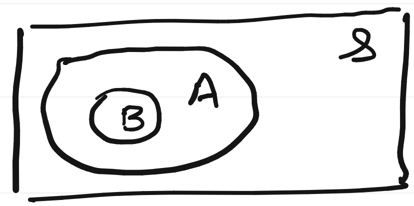
Given $(\Omega, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$, knowing that B has occured may tell us something about whether or not A has occured

Ex. 1



If B occured, A did not

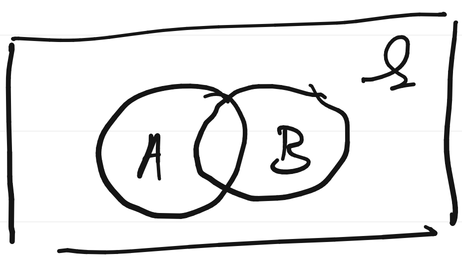
Ex. 2



$B \subset A$

If B occured A occured,

Ex. 3



In general, knowing B has occured may change your belief that A has occured.

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$, the <u>conditional probability of A conditioned on B</u> ("A given B") is

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)},$$

assuming $P(B) \neq 0$.

n.b. If $P(B) = 0$, then $P(A \cap B) = 0$, and this leaves us with $P(A|B) = \frac{0}{0}$ which is undefined.

6.14

n.b. 1. If $A \subset B$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \geq P(A)$$

2. If $B \subset A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3. If $A \cap B = \phi$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = \frac{0}{P(B)} = 0$$

4. $$P(A|B) = \frac{P(A \cap B)}{P(B)} \geq P(A \cap B)$$

$$0 < P(B) \leq 1$$

Fact: If $P(\cdot)$ (from $(\mathcal{S}, \mathcal{F}, P)$) is a valid probability measure, then $P(\cdot \mid B)$ is also a valid probability measure for any $B \in \mathcal{F}$ such that $P(B) \neq 0$.

Proof: (exercise) Verify the axioms of probability hold for $P(\cdot \mid B)$.

$$(\mathcal{S}, \mathcal{F}, P) \xRightarrow{\text{B has occurred}} (\mathcal{S}, \mathcal{F}, P(\cdot \mid B))$$

n.b. $(\mathcal{S}, \mathcal{F}, P(\cdot \mid B))$ is a valid prob. space because $(\mathcal{S}, \mathcal{F}, P)$ is a valid prob. space.