

Session 8

Exam 1: Tuesday February 6, 2024

Closed Book, Closed Notes,
no Crib Sheets, no calculators

- On Campus FNY Section:
Here, during Session 9 at 10:30 - 11:45 am
- On Campus Overflow Section:
Room ARMS 1028 at 10:30 - 11:45 am
(TA Brad Fitzgerald will proctor.)
- Distance EPE Section:
See instructions from Lynn Hegewald
(Using Brightspace/Eximity)

Recall...

Independent Experiments

8.1

Sometimes, the outcomes of the two constituent experiments are unrelated:

$$A \times \mathcal{Q}_2 \perp\!\!\!\perp \mathcal{Q}_1 \times B, \quad \begin{array}{l} \forall A \in \mathcal{F}_1 \\ \forall B \in \mathcal{F}_2 \end{array}$$

↑
independent

In this case we say that the two experiments $(\mathcal{Q}_1, \mathcal{F}_1, \mathcal{P}_1)$ and $(\mathcal{Q}_2, \mathcal{F}_2, \mathcal{P}_2)$ are independent experiments.

Recall...

8.2

For independent experiments, we assign the probability $P(\cdot)$ as

$$\begin{aligned} P(A \times B) &= P((A \times \Omega_2) \cap (\Omega_1 \times B)) \\ &= P(A \times \Omega_2) \cdot P(\Omega_1 \times B) \\ &= P_1(A) \cdot P_2(B) \end{aligned}$$

The axioms of probability fill in the probabilities of events that cannot be written as cartesian products (but can be written as a union of disjoint cartesian products.)

So for two independent experiments

$(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$

we have a combined experiment

(Ω, \mathcal{F}, P) with

$$\Omega = \Omega_1 \times \Omega_2$$

$$\mathcal{F} = \sigma(\{A \times B : \forall A \in \mathcal{F}_1, \text{ and } \forall B \in \mathcal{F}_2\})$$

$$P(A \times B) = P_1(A) P_2(B)$$

Axioms of probability fill in
everything else.

Generalization to n independent experiments

8.4

• Let $(\Omega_1, \mathcal{F}_1, P_1), (\Omega_2, \mathcal{F}_2, P_2), \dots, (\Omega_n, \mathcal{F}_n, P_n)$
be n random experiments

• Form the combined experiment (Ω, \mathcal{F}, P)

with
$$\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$$

$$\mathcal{F} = \sigma(\{A_1 \times A_2 \times \dots \times A_n : \forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2, \dots, \forall A_n \in \mathcal{F}_n\})$$

If the experiments are independent, then

$$P(A_1 \times A_2 \times \dots \times A_n) = P_1(A_1) \cdot P_2(A_2) \cdot \dots \cdot P_n(A_n)$$
$$\forall A_1 \in \mathcal{F}_1, \forall A_2 \in \mathcal{F}_2, \dots, \forall A_n \in \mathcal{F}_n$$

Then the axioms of probability fill in the rest.

Important Special Case: Bernoulli Trials

8.5

- Consider a simple experiment $(\mathcal{S}_0, \mathcal{F}_0, P_0)$.
- Now assume I independently perform this experiment n times:

$$(\mathcal{S}_1, \mathcal{F}_1, P_1), (\mathcal{S}_2, \mathcal{F}_2, P_2), \dots, (\mathcal{S}_n, \mathcal{F}_n, P_n),$$

where

$$\mathcal{S}_1 = \mathcal{S}_2 = \dots = \mathcal{S}_n \equiv \mathcal{S}_0$$

$$\mathcal{F}_1 = \mathcal{F}_2 = \dots = \mathcal{F}_n \equiv \mathcal{F}_0$$

$$P_1(\cdot) = P_2(\cdot) = \dots = P_n(\cdot) \equiv P_0(\cdot)$$

We want to form the combined experiment of these n independent trials.

Our combined experiment:

8.6

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_n$$

$$\mathcal{F} = \sigma(\{\text{all cylinder sets}\})$$

Now suppose I have an event $A \in \mathcal{F}_0$ having probability $P_0(A) = p$, $0 \leq p \leq 1$.

When I repeat the experiment n times, I want to know the probability of $B_k \in \mathcal{F}$ defined as

$B_k \stackrel{\Delta}{=} A$ occurs exactly k times
in the n repetitions of
($\mathcal{S}_0, \mathcal{F}_0, P_0$)

Let's write the probability of B_k as

$$P_n(k) \triangleq P(B_k).$$

Theorem: $P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \dots, n$,
where $p = P_0(A)$.

Proof: See Papoulis

We sketch the proof as follows...

Idea: There are exactly $\binom{n}{k}$ sequences of outcomes of the simple experiment where A occurs exactly k times and \bar{A} occurs $n-k$ times.

e.g. $n=4, k=2 \quad \binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2!} = 6$

$AA\bar{A}\bar{A}, A\bar{A}A\bar{A}, \bar{A}AA\bar{A}$
 $\bar{A}\bar{A}AA, \bar{A}A\bar{A}A, A\bar{A}\bar{A}A$

- Each sequence corresponds to a unique $E_j \in \mathcal{F}$

- Each event E_j has probability

$$P(E_j) = p^k (1-p)^{n-k}$$

- The events E_j are disjoint.

$$\begin{aligned} \therefore p_n(k) &= P(B_k) = P(E_1 \cup E_2 \cup \dots \cup E_{\binom{n}{k}}) \\ &= P(E_1) + P(E_2) + \dots + P(E_{\binom{n}{k}}) \\ &= \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$



Example: A box contains two coins, a fair coin and a "strange" coin.

- The fair coin has a probability $\frac{1}{2}$ of coming up "Heads" when tossed.
- The "strange" coin has a probability $\frac{3}{4}$ of coming up "Heads" when tossed.
- I select a coin at random from the box and flip it 4 times. It comes up "Heads" three times.

Q: What is the probability I selected the "strange" coin?

$$\mathcal{S}_1 = \{F, S\}$$

$$\mathcal{S}_2 = \{HHHH, HHHT, HHTH, \dots, TTTT\}$$

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \quad \left(\begin{array}{l} \text{I select a coin,} \\ \text{and I flip it 4 times} \end{array} \right)$$

Let F = fair coin selected

S = strange coin selected

A_3 = 3 "Heads" in 4 tosses

What is $P(S|A_3)$?

$$P(S|A_3) = \frac{P(S \cap A_3)}{P(A_3)} = \frac{P(A_3|S)P(S)}{P(A_3)}$$

8.12

Assumption $P(S) = \frac{1}{2}$

$$P(A_3|S) = \binom{4}{3} \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^1 = 4 \left(\frac{27}{64}\right) \left(\frac{1}{4}\right) = \frac{27}{64}$$

$$P(A_3) = P(A_3 \cap \Omega) = P(A_3 \cap (S \cup F))$$

$$= P((A_3 \cap S) \cup (A_3 \cap F))$$

disjoint

$$= P(A_3 \cap S) + P(A_3 \cap F)$$

$$= P(A_3|S)P(S) + P(A_3|F)P(F)$$

$$= \frac{27}{64} \cdot \frac{1}{2} + \binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 \cdot \frac{1}{2}$$

$$= \frac{27}{128} + \frac{1}{8} = \frac{43}{128}$$

Thus we have

8.13

$$P(S|A_3) = \frac{P(A_3|S)P(S)}{P(A_3)}$$

$$= \frac{\left(\frac{27}{64}\right) \cdot \left(\frac{1}{2}\right)}{\frac{43}{128}} = \frac{27}{43} \approx 0.6279$$

n.b. $P(F|A_3) = \frac{P(A_3|F)P(F)}{P(A_3)} = \dots = \frac{16}{43} \approx 0.3721$

$$P(\cdot|A_3) \quad F = \bar{S}$$

$$P(\bar{S}|A_3) = 1 - P(S|A_3)$$

Classical Probability

8.14

In classical probability, we have (Ω, \mathcal{F}, P) in which

- Ω is finite: $|\Omega| = n$
- $\mathcal{F} = \mathcal{P}(\Omega)$: $|\mathcal{F}| = 2^n$
- All outcomes are equally likely, which means we have pmf
$$p(\omega) = \frac{1}{n}, \quad \forall \omega \in \Omega$$

$$\Rightarrow P(A) = \sum_{\omega \in A} p(\omega) = \frac{|A|}{n} = \frac{|A|}{|\Omega|}$$

So in classical probability, we have

8.15

$$\bullet P(\emptyset) = \frac{|\emptyset|}{|\Omega|} = \frac{0}{n} = 0.$$

$$\bullet P(\Omega) = \frac{|\Omega|}{|\Omega|} = \frac{n}{n} = 1$$

• If $A, B \in \mathcal{F}$ are disjoint (i.e. $A \cap B = \emptyset$)
then

$$P(A \cup B) = \frac{|A \cup B|}{|\Omega|} = \frac{|A| + |B|}{|\Omega|}$$

$$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = P(A) + P(B).$$

What about $A, B \in \mathcal{F}$ that are
not disjoint?

8.16

$$\begin{aligned} P(\underline{A \cup B}) &= \frac{|A \cup B|}{|\mathcal{S}|} = \frac{|A| + |B| - |A \cap B|}{|\mathcal{S}|} \\ &= \frac{|A|}{|\mathcal{S}|} + \frac{|B|}{|\mathcal{S}|} - \frac{|A \cap B|}{|\mathcal{S}|} \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

This extends to general finite unions:

8.17

If $A_1, A_2, \dots, A_n \in \mathcal{F}$ and they are disjoint, then

$$A_i \cap A_j = \emptyset, \quad \forall i \neq j$$

Then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i|$$

\Rightarrow For classical probability, if A_1, \dots, A_n are disjoint, then

$$P\left(\bigcup_{i=1}^n A_i\right) = \frac{\left|\bigcup_{i=1}^n A_i\right|}{|\Omega|} = \sum_{i=1}^n \frac{|A_i|}{|\Omega|} = \sum_{i=1}^n P(A_i)$$

Note also in classical probability
that since

8.18

$$|\bar{A}| = |\Omega| - |A|$$

$$P(\bar{A}) = \frac{|\bar{A}|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|}$$

$$= 1 - \frac{|A|}{|\Omega|} = 1 - P(A).$$

So the classical probability measure behaves like a general axiomatic prob. measure.

It can be argued that Kolmogorov selected the axioms of probability based on how classical probability behaves.

Exam 1
cut-off

Exam 1 covers Homework
Assignments 1-3,
Lectures 1-8.