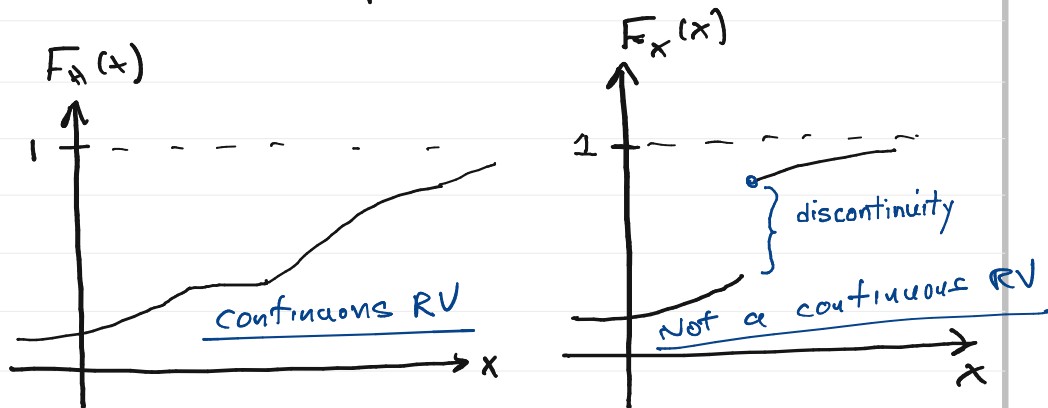


## Session 12

Recall...

Defn: We say that a random variable is (absolutely) continuous if  $F_X(x)$  is a continuous function at all points  $x \in \mathbb{R}$ .

12.1

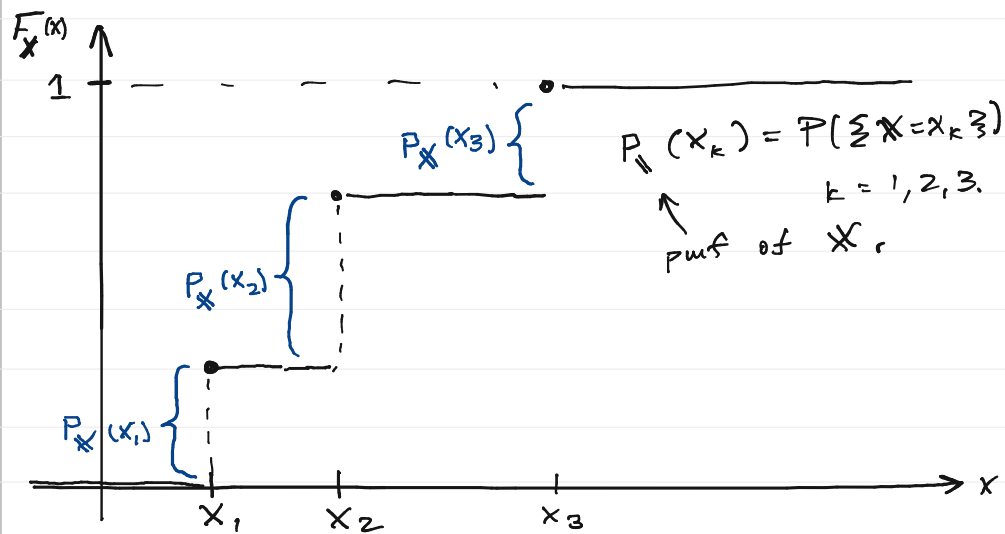


Recall...

We say that a RV  $X$  is discrete if it takes on values from a discrete (finite or countable) subset of  $\mathbb{R}$ .

12.2

In this case,  $F_X(x)$  is a "staircase function".



Defn: The probability density function of a RV  $X$  is defined as the derivative of the cdf  $F_X(x)$  w.r.t.  $x$ :

12.3

$$f_X(x) = \frac{dF_X(x)}{dx}$$

n.b. (i)  $f_X(x) \geq 0$ ,  $\forall x \in \mathbb{R}$ .

$$\begin{aligned} \text{(ii)} \quad \int_{-\infty}^{\infty} f_X(x) dx &= F_X(\infty) - F_X(-\infty) \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

n.b. We will broaden our "definition" of derivative to include the derivative of a step discontinuities as Dirac  $\delta$ -functions.

12.4

Dirac  $\delta$ -functions :  $\delta(x)$

$$(i) \quad \delta(x) = 0, \quad \forall x \neq 0.$$

$$(\delta(x-x_0) = 0, \quad \forall x \neq x_0)$$

$$(ii) \quad \int_{-\infty}^{\infty} \delta(x) dx = \int_{-\epsilon}^{\epsilon} \delta(x) dx = 1, \quad \forall \epsilon > 0$$

These two defining properties give rise to the "sifting property" of the Dirac delta function :

12.5

$$\begin{aligned} \int_{-\infty}^{\infty} g(x) \delta(x-x_0) dx &= \int_{-\infty}^{\infty} g(x_0) \delta(x-x_0) dx \\ &= g(x_0) \int_{-\infty}^{\infty} \delta(x-x_0) dx = g(x_0) \cdot 1 \\ &= g(x_0) \end{aligned}$$

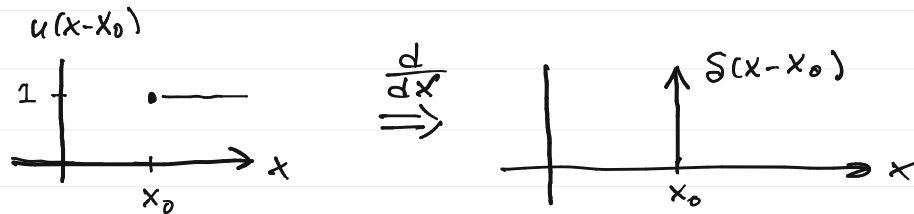
$$\therefore \int_{-\infty}^{\infty} g(x) \delta(x-x_0) dx = g(x_0)$$

The sifting property of the Dirac  $\delta$ -fun-

Suppose I have

12.6

$$u(x-x_0) \triangleq \mathbb{1}_{[x_0, \infty)}(x) = \begin{cases} 1, & x \in [x_0, \infty) \\ 0, & x \in (-\infty, x_0) \end{cases}$$



$$\left\| \frac{d}{dx} u(x-x_0) \right\| = \delta(x-x_0)$$

$$u(x) = \int_{-\infty}^x \delta(r-x_0) dr = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0 \\ 1, & x = x_0 \end{cases}$$

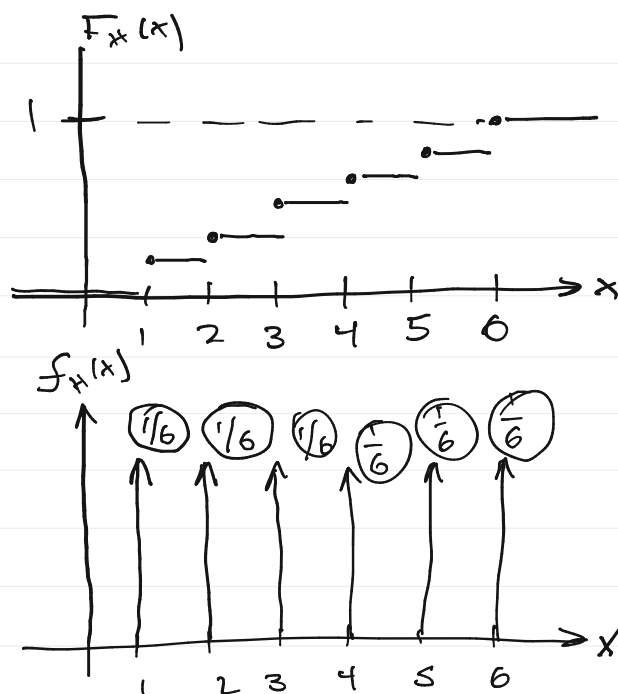
Example: Consider a RV that is the numerical outcome of rolling a fair die.

12.7

$$F_{\#}(x) = P(\{X \leq x\})$$

$$= \frac{1}{6} \cdot \mathbb{1}_{[1, \infty)}(x) + \frac{1}{6} \mathbb{1}_{[2, \infty)}(x) + \dots + \frac{1}{6} \mathbb{1}_{[6, \infty)}(x)$$

$$f_{\#}(x) = \left\| \frac{dF_{\#}(x)}{dx} \right\| = \frac{1}{6} \delta(x-1) + \frac{1}{6} \delta(x-2) + \dots + \frac{1}{6} \delta(x-6)$$



### Properties of the pdf of a RV

$$1. f_X(x) \geq 0, \forall x \in \mathbb{R}$$

$$2. F_X(x) = \int_{-\infty}^x f_X(\alpha) d\alpha$$

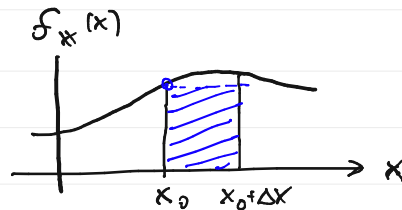
$$3. \int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$4. P(\{x_1 \leq X \leq x_2\}) = \int_{x_1}^{x_2} f_X(x) dx$$

For a continuous RV  $X$

12.10

$$P(\{x_0 < X \leq x_0 + \Delta x\}) = \int_{x_0}^{x_0 + \Delta x} f_X(x) dx$$
$$\approx f_X(x_0) \cdot \Delta x, \text{ for small } \Delta x$$



Recall:  $f_X(x) \approx \frac{F_X(x + \Delta x) - F_X(x)}{\Delta x}$

$$\Rightarrow f_X(x) \cdot \Delta x \approx F_X(x + \Delta x) - F_X(x)$$
$$= P(\{x < X \leq x_0 + \Delta x\})$$

12.11

We often describe a RV  $X$   
by specifying its cdf or  
pdf and completely ignoring  
the underlying  $(\mathcal{S}, \mathcal{F}, P)$ .

(The underlying  $(\mathcal{S}, \mathcal{F}, P)$  is there,  
but we just don't think about it.)

Ex. 1 Gaussian RV

12.12

A RV  $X$  is Gaussian if it has a pdf of the form

$$f_{\#}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \forall x \in \mathbb{R}$$

where  $\mu \in \mathbb{R}$  and  $\sigma > 0$ .

n.b.  $F_{\#}(x) = \int_{-\infty}^x f_{\#}(\alpha) d\alpha = \Phi\left(\frac{x-\mu}{\sigma}\right)$   $\left(G\left(\frac{x-\mu}{\sigma}\right)\right)$   
in Papoulis  $\mathcal{N}[\mu, \sigma^2]$

where  $\Phi(r) = \int_{-\infty}^r \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

12.13

$\Phi(\cdot)$  cannot be written in "closed form". It can be numerically computed and is widely tabulated.

So if  $X$  is a Gaussian RV with  $\mu \in \mathbb{R}$  and  $\sigma > 0$ , then

$$\begin{aligned} P(\{a < X \leq b\}) &= F_{\#}(b) - F_{\#}(a) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$

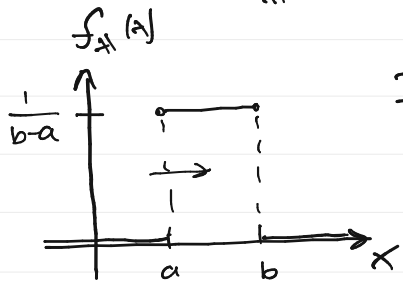
## Ex.2 Uniformly Distributed RV

12.14

A RV has a uniform distribution,

$$X \sim U[a, b], \quad a < b$$

$$f_X(x) = \frac{1}{b-a} \cdot \mathbb{1}_{[a,b]}(x)$$



Integrate  
 $\Rightarrow$

