

Session 28

ECE 600 Final Exam

Monday, April 29, 2024

3:30 to 5:30 pm

Room FNY G140 { For FNY and
ONC (overflow) Sections

This will be a comprehensive exam.

This is a closed book, closed notes exam.

You may not use a calculator.

Start each problem on a new page.

- 10 "simple" multiple choice* questions 50%

* There will be room to show your work.

- Two "work-out" problems 34%

- One page of True/False Questions 16%

EPE Online Section Answer Form

ECE600 Random Variables and Waveforms
Spring ~~2022~~ 2024

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MSEE 336

Final Exam Online Section Answer Form

~~May 4, 2022~~
April 29, 2024

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

$\delta(\tau) \leftrightarrow 1$ $e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega - \beta)$ $e^{-\alpha \tau } \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$	$1 \leftrightarrow 2\pi\delta(\omega)$ $\cos \beta\tau \leftrightarrow \pi\delta(\omega - \beta) + \pi\delta(\omega + \beta)$ $e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$
$e^{-\alpha \tau } \cos \beta\tau \leftrightarrow \frac{\alpha}{\alpha^2 + (\omega - \beta)^2} + \frac{\alpha}{\alpha^2 + (\omega + \beta)^2}$ $2e^{-\alpha\tau^2} \cos \beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}} [e^{-(\omega - \beta)^2/4\alpha} + e^{-(\omega + \beta)^2/4\alpha}]$	$\begin{cases} 1 - \frac{ \tau }{T} & \tau < T \\ 0 & \tau > T \end{cases} \leftrightarrow \frac{4 \sin^2(\omega T/2)}{T\omega^2}$ $\frac{\sin \alpha\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & \omega < \alpha \\ 0 & \omega > \alpha \end{cases}$

1-10	
11	
12	
13	
Total	

Name: _____

ID #: _____

Directions:

1. **Print** your name and student number on the cover page.
2. Exam is closed book, closed notes, and no calculators.
3. Clearly designate all answers asked for (arrows, underline, box, etc.)

Problems 1-10 are multiple choice problems worth 5 points each. For each problem, write the letter corresponding to the best answer next to the problem number. Space is provided to work out your solution for each of these problems. Please show your work! If your final grade is near a borderline, the quality of your written solutions could significantly impact your final grade.

1. Answer:

C

$$\begin{aligned}
 \Phi^*(\omega) &= E[e^{i\omega X}] \\
 &= \sum_{k=0}^n e^{i\omega k} \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \sum_{k=0}^n \binom{n}{k} (pe^{i\omega})^k (1-p)^{n-k} \\
 &\stackrel{\text{Binomial Thm.}}{=} (pe^{i\omega} + 1 - p)^n \\
 &= (1 + p(e^{i\omega} - 1))^n \\
 &\text{etc.}
 \end{aligned}$$

2. Answer:



10. Answer:

11. Problems 11 is made up of 8 True/False questions, worth 2 points each. Fill in your answers T (true) or F (false) below, corresponding to the statements A-H in problem 11 on the exam.

A. _____

B. _____

C. _____

D. _____

E. _____

F. _____

G. _____

H. _____

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Name: _____

Problems 12 and 13 are "work out" problems for which partial credit will be awarded for correctly reasoned work. It is important that you coherently present your thinking in the solution of these problems if you wish to receive partial credit (or full credit for that matter.) Please work problems 12 and 13 of the exam in the designated space below.

12. Problem 12 Solution:

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Name: _____

(Problem 12 Solution Continued)

13. Problem 13 Solution:

(Problem 13 Solution Continued)

Stochastic Processes

28.1

- The idea of a stochastic process is a straightforward extension of the idea of a random variable.
- Instead mapping each outcome $w \in \Omega$ to a number $X(w)$, we map it to a function of time $X(t, w)$

28.1

Stochastic Processes

28.2

Defn⁽¹⁾: A random process defined on a random experiment (Ω, \mathcal{F}, P) is a family $\{X(t); t \in \mathbb{R}\}$ of random variables defined on (Ω, \mathcal{F}, P) and indexed by t .

Recall: A RV $X: \Omega \rightarrow \mathbb{R}$ for a particular outcome $w_0 \in \Omega$ is just a real number $X(w_0)$.

A random process $X(t, w)$ is a function of two variables: time and the outcome $w \in \Omega$.

Random process

28.3.

$$X(t, \omega) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

↑
real time
values

↑
outcomes of
a random
experiment

Just as we often write the random variable $X(\omega)$ as X (hiding explicit dependence on $\omega \in \Omega$)

so we often write the random process $X(t, \omega)$ as $X(t)$. (again hiding the explicit dependence on $\omega \in \Omega$).

The boldface notation X or $X(t)$ tells us there is an implicit dependence on $\omega \in \Omega$.

2

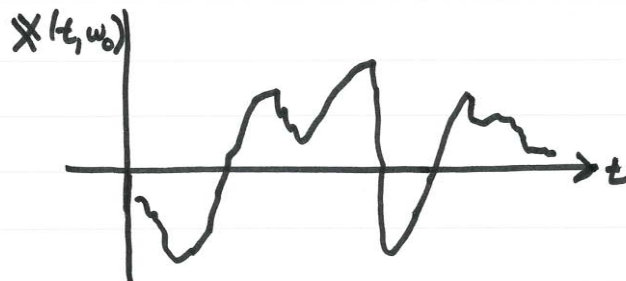
28.3

If we fix a particular $\omega_0 \in \Omega$ then we have

28.4

$$X(\cdot, \omega_0) : \mathbb{R} \rightarrow \mathbb{R}$$

↑
 t



For each $\omega \in \Omega$, $X(\cdot, \omega)$ is a different function of time.

3

Defn : For a fixed outcome $\omega_0 \in \Omega$, 28.5
the time function $X(\cdot) = X(\cdot, \omega_0)$
is called the sample path of
the random process $X(t)$ corresponding
to ω_0 . Each $\omega \in \Omega$ has a
sample path $X(\cdot) = X(\cdot, \omega)$.
The set of all sample paths

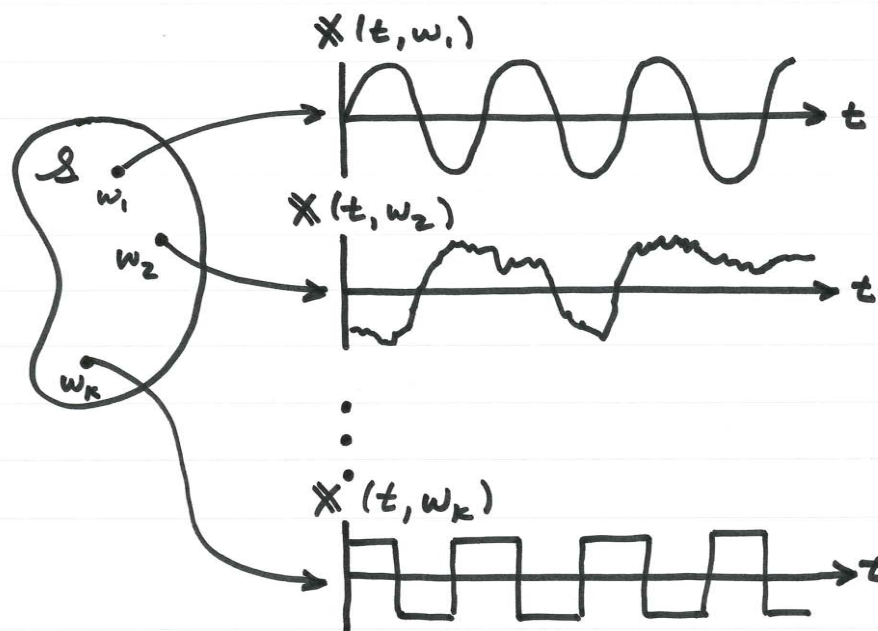
$\mathcal{E} = \{X(\cdot) : X(\cdot) = X(\cdot, \omega), \text{ for } \omega \in \Omega\}$
is called the ensemble of
the R.P. $X(t)$.

Having established these ideas, we can give
the following equivalent definition of a
random process ...

Defn⁽²⁾ : Consider a random 28.6
experiment (Ω, \mathcal{F}, P) . A
function $X : \Omega \rightarrow \mathcal{E}$, where \mathcal{E}
is a set of functions of
time (sample paths) is called
a random process. That is,
 X is a mapping assigning
a function of time $X(\cdot, \omega)$
to each $\omega \in \Omega$.

A picture appears as follows:

28.7



6

Example: Roll a die:

28.8

$$S = \{ \underset{1}{\omega_1}, \underset{2}{\omega_2}, \underset{3}{\omega_3}, \underset{4}{\omega_4}, \underset{5}{\omega_5}, \underset{6}{\omega_6} \}$$

We know how to construct (S, \mathcal{F}, P) .

Now construct a random process $X(t)$ as

$$X(t, \omega_k) = \cos(2\pi k t), \quad k = 1, 2, \dots, 6.$$

\uparrow numerical outcome on die






So we now have a random process $X(t)$ that is a cosine function whose frequency is selected at random:

7

(example continued)

28.9

$$X(t, \omega_k) = \cos(2\pi k t)$$

ω	$X(t, \omega)$	
ω_1	$\cos 2\pi t$	
ω_2	$\cos 4\pi t$	
ω_3	$\cos 6\pi t$	
ω_4	$\cos 8\pi t$	
\vdots		
ω_6	$\cos 12\pi t$	

Note that once we know the particular $\omega_k \in \mathcal{S}$ that is the outcome of the experiment, there is nothing random about the random process.

28.10

Randomness only enters into the random process through the selection of $\omega \in \mathcal{S}$ in the random experiment.

Note: A random process is completely analogous to a Random variable:

Random Variable $X: \mathcal{S} \rightarrow \mathbb{R}$ (real line)

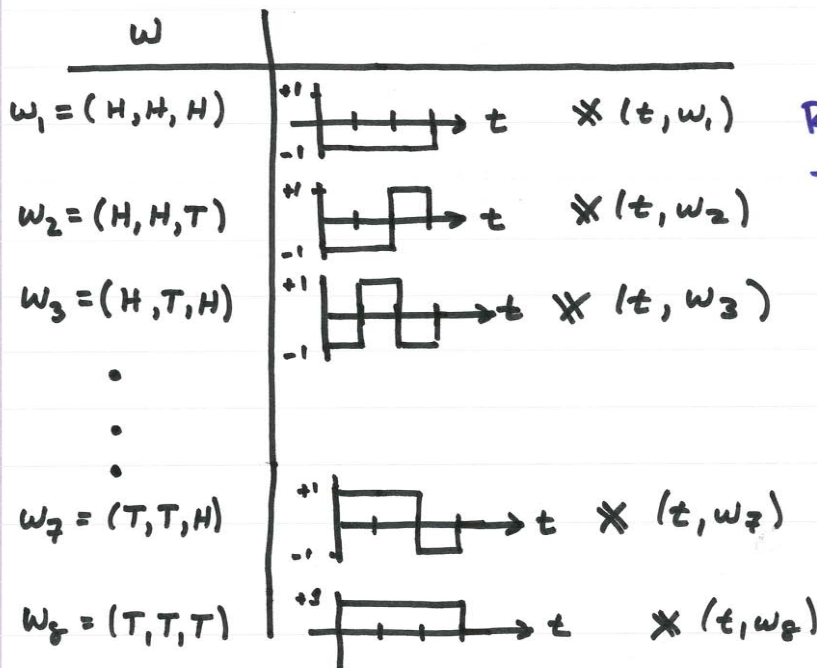
Random Process: $X(t): \mathcal{S} \rightarrow \mathcal{E}$ (set of functions of time)

Example: I flip a coin three times (once a second) and construct a random process by making a waveform take on the value during that second corresponding to the outcome of the coin during that second:

$$\begin{aligned} H &\rightarrow -1 \\ T &\rightarrow +1 \end{aligned} \quad (*)$$

I can consider this to be a randomized experiment with $2^3 = 8$ possible outcomes:

$$\mathcal{S} = \{w_1, \dots, w_8\}$$

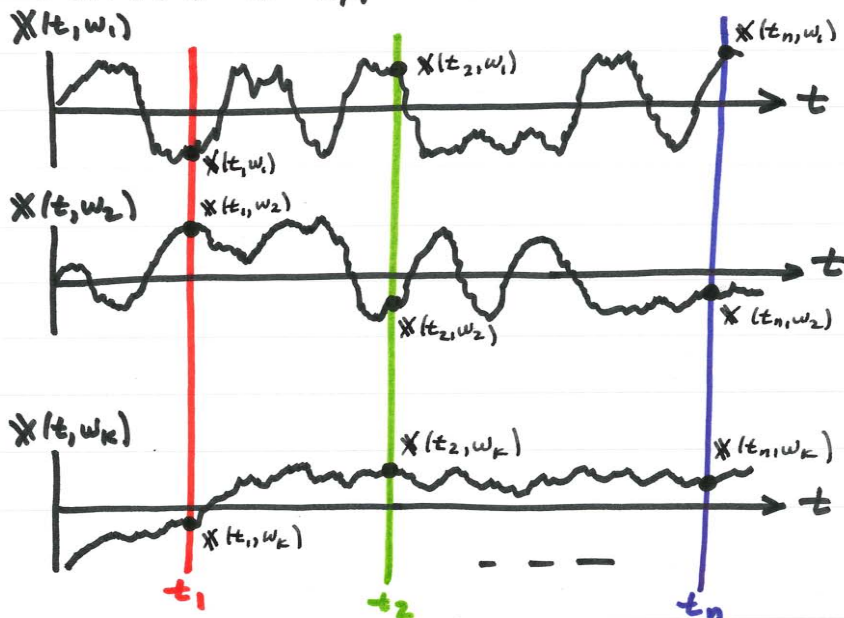


Randomness enters in only in the determination of w by flipping the coin 3 times.

Once you know w , $* (t, w)$ is specified by relation $(*)$ non-randomly.

Note: If we evaluate a random process at time t_i , we get a random variable $X(t_i)$:

28.13



If we sample the random process $X(t)$ at n points in time, we get n j -dist RVs $X(t_1), X(t_2), \dots, X(t_n)$.

Defn: Fix n time instants

28.14

t_1, \dots, t_n . Then the n -th order cdf of $X(t)$ at times t_1, t_2, \dots, t_n is the joint cdf of the n j -dist RVs $X(t_1), X(t_2), \dots, X(t_n)$:

$$F_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n) = P(\{X(t_1) \leq x_1\} \cap \dots \cap \{X(t_n) \leq x_n\})$$

$$= "F_{X(t)}(x_1, \dots, x_n; t_1, \dots, t_n)"$$

and the corresponding n -th order pdf of $X(t)$ is

$$f_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n) = \frac{\partial^n F_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$$

$$= "f_{X(t)}(x_1, \dots, x_n; t_1, \dots, t_n)"$$

Theorem: The probabilistic behavior of a random process $X(t)$ is completely characterized by the collection of all n -th order cdfs or pdfs, for all t_1, t_2, \dots, t_n , and all $n = 1, 2, 3, \dots$ (all $n \in \mathbb{N}$.)

28.15

Proof: (Kolmogorov) Way beyond this course
But we can use the fact.

2

Stationary Random Processes

28.16

"Stationary" \triangleq staying in one place throughout time.

What if the n -th order distributions don't change with a shift in the time origin?

"Probabilistic behavior is the same yesterday, today and tomorrow."

Probabilistic description "stays in one place" throughout time.

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28.16

Defn: A random process $X(t)$ is called 28.17
n-th order stationary if all n-th order
 cdfs or pdfs are invariant to time
 shifts in the origin:

$$f_{X(t_1) \dots X(t_n)}(x_1, \dots, x_n) = f_{X(t_1+c) \dots X(t_n+c)}(x_1, \dots, x_n)$$

for all $c \in \mathbb{R}$ and for any n
 time instants t_1, \dots, t_n .

Defn: A random process $X(t)$ is called
stationary, or strict sense stationary
 (SSS), if $X(t)$ is n-th order
 stationary for all $n=1, 2, 3, \dots$

Note: If $X(t)$ is stationary, 28.18
 it is first-order ($n=1$) stationary.

$$\begin{aligned} \Rightarrow f_{X(t_1)}(x) &= f_{X(t_1+c)}(x), \forall c \in \mathbb{R} \\ &= f_{X(t_2)}(x), \text{ for any } t_2 \in \mathbb{R} \\ &= f(x) \quad (\text{not a function of time}) \end{aligned}$$

$$\begin{aligned} \therefore E[X(t)] &= \int_{-\infty}^{\infty} x f_{X(t)}(x) dx \\ &= \int_{-\infty}^{\infty} x f(x) dx = \text{constant} \end{aligned} \quad \left(\begin{array}{l} \text{Not a} \\ \text{function} \\ \text{of} \\ \text{time.} \end{array} \right)$$

Similarly

$$\begin{aligned} \text{var}(X(t)) &= E[X^2(t)] - (E[X(t)])^2 \\ &= \sigma^2 = \text{constant} \quad \left(\begin{array}{l} \text{Not a function} \\ \text{of time.} \end{array} \right) \end{aligned}$$

Example: non-stationary random process.

28.19

$$\text{Let } X(t) \triangleq e^{-\Lambda t} \cdot \mathbb{1}_{[0, \infty)}(t)$$

where Λ is a uniformly distributed RV on $[0, 1]$.

$X(t)$ is a random process.

At $t = t_1$, we have that $X(t_1)$ is a RV:

$$\begin{aligned} X(t_1) &= e^{-\Lambda t_1} \cdot \mathbb{1}_{[0, \infty)}(t_1) = g(\Lambda, t_1) \\ &= g_{t_1}(\Lambda). \end{aligned}$$

So

28.20

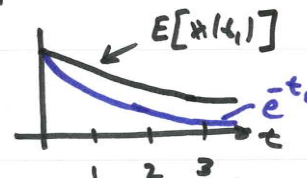
$$E[X(t_1)] = E[g_{t_1}(\Lambda)]$$

$$= \int_{-\infty}^{\infty} g_{t_1}(a) f_{\Lambda}(a) da$$

$$= \int_0^1 g_{t_1}(a) \cdot \mathbb{1} da = \int_0^1 e^{-at_1} da \cdot \mathbb{1}_{[0, \infty)}(t_1)$$

$$= \mathbb{1}_{[0, \infty)}(t_1) \left. \frac{e^{-at_1}}{-t_1} \right|_{a=0}^1 = \frac{1 - e^{-t_1}}{t_1} \cdot \mathbb{1}_{[0, \infty)}(t_1)$$

$$\therefore E[X(t_1)] = \frac{1 - e^{-t_1}}{t_1} \cdot \mathbb{1}_{[0, \infty)}(t_1)$$



Because $E[X(t)]$ is a function of t , it follows that $f_{X(t)}(x)$ must also be a function of t (you can find $f_{X(t)}(x)$ as an exercise)

$\Rightarrow X(t)$ is not first order stationary

$\Rightarrow X(t)$ is not stationary

In the special case of 2nd order ($n=2$) stationary processes, we have

$$f_{X(t_1), X(t_2)}(x_1, x_2) = f_{X(t_1+c), X(t_2+c)}(x_1, x_2), \quad \forall c \in \mathbb{R}.$$

This means that the second-order pdf

$$f_{X(t_1), X(t_2)}(x_1, x_2) = "f_{X}(x_1, x_2; t_1, t_2)"$$

can depend on t_1 and t_2 only through the time difference $t_1 - t_2$.

In such cases, the correlations between $X(t_1)$ and $X(t_2)$ can depend only on the time difference $t_1 - t_2$.

This motivates the following weaker form of stationarity:

28.23

Defn: A random process $X(t)$ is called wide-sense stationary (WSS) if

$$(i) E[X(t)] = \bar{X} = \text{constant}$$

and

$$(ii) E[X(t_1) \cdot X(t_2)] = R(t_1 - t_2)$$

i.e., it depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

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Note:

1. If a random process $X(t)$ is first and second order ($n=1,2$) stationary, and if $E[X(t)]$ and $E[X(t_1)X(t_2)]$ exist, then $X(t)$ is wide-sense stationary. The converse is not true. 28.24
2. If a R.P. $X(t)$ is stationary (S.S.S), it is also wide-sense stationary (W.S.S.) if $E[X(t_1)]$ and $E[X(t_1)X(t_2)]$ exist.
3. If $X(t)$ is a W.S.S R.P., $E[X(t_1)X(t_2)]$, $\text{cov}(X(t_1), X(t_2))$, and the correlation coefficient $r_{X(t_1)X(t_2)}$ will depend only on the time difference $t_1 - t_2$.

Example: Consider a R.P.

28.25

$$X(t) = \cos(\omega_0 t + \Theta)$$

where Θ is a R.V. uniformly distributed on $[0, 2\pi)$:

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & , 0 \leq \theta < 2\pi \\ 0 & , \text{elsewhere} \end{cases}, \omega_0 = \text{constant} \\ \text{(radian frequency, not outcome)}$$

Is $X(t)$ W.S.S? Let's check the two defining conditions

24

(i) $E[X(t)] = E[\cos(\omega_0 t + \Theta)]$

28.26

$$= E[\underbrace{\cos(\omega_0 t)} \cos(\Theta) - \underbrace{\sin(\omega_0 t)} \sin(\Theta)]$$

$$= \int_{-\infty}^{\infty} (\cos(\omega_0 t) \cos \theta - \sin(\omega_0 t) \sin(\theta)) f_{\Theta}(\theta) d\theta$$

$$= \frac{\cos \omega_0 t}{2\pi} \int_0^{2\pi} \cos \theta d\theta - \frac{\sin \omega_0 t}{2\pi} \int_0^{2\pi} \sin \theta d\theta$$

$$= 0 \quad \therefore E[X(t)] = 0 = \text{constant}. \quad \square$$

n.b. Even easier:

$$E[X(t)] = E[\cos(\omega_0 t + \Theta)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega_0 t + \theta) d\theta$$



Integral over one period is 0.

$$\cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \quad 28.27$$

$$\begin{aligned} \text{(ii)} : E[X(t_1)X(t_2)] &= E[\cos(\omega_0 t_1 + \Theta) \cdot \cos(\omega_0 t_2 + \Theta)] \\ &= E\left[\frac{1}{2} (\cos(\omega_0(t_1+t_2) + 2\Theta) + \cos(\omega_0(t_1-t_2)))\right] \\ &= \frac{1}{2} E[\cos(\omega_0(t_1+t_2) + 2\Theta)] + \frac{1}{2} E[\cos(\omega_0(t_1-t_2))] \\ &= \frac{1}{2} \cos(\omega_0(t_1-t_2)) \quad \checkmark \end{aligned}$$

$$\therefore \text{(i)} \quad E[X(t)] = 0 = \text{constant}, \quad 28.28$$

$$\text{(ii)} \quad E[X(t_1)X(t_2)] = \frac{1}{2} \cos \omega_0(t_1-t_2).$$

$\therefore X(t)$ is a W.S.S. R.P.