Session 6

Recall...

6.1

Example of a set A where the Riemann integral does not exist

Suppose I have a pdf

$$f(r) = 1_{[0,1]}(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $A = \mathbb{Q}$ = rational numbers.

$$P(A) = P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r) \cdot 1_{\mathbb{Q}}(r)\, dr$$

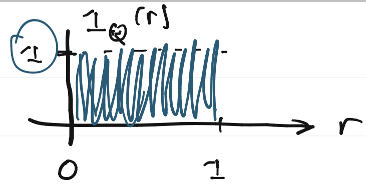
$$= \int_0^1 1_{\mathbb{Q}}(r)\, dr$$

does not exist as a Riemann integral

Recall...

6.2

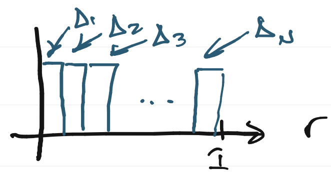
$$P(\mathbb{Q}) = \int_{-\infty}^{\infty} f(r)\cdot \mathbb{1}_{\mathbb{Q}}(r)\,dr$$



$$= \int_0^1 \mathbb{1}_{\mathbb{Q}}(r)\,dr$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} |\Delta_k|\cdot \mathbb{1}_{\mathbb{Q}}(\hat{x}_k)$$

$$\hat{x}_k \in \Delta_k$$



$$\underline{\text{sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\min_{x\in\Delta_k} \mathbb{1}_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 0 = 0$$

$$\overline{\text{Sum}}_N = \sum_{k=1}^{N} |\Delta_k|\cdot\left(\max_{x\in\Delta_k} \mathbb{1}_{\mathbb{Q}}(x)\right) = \sum_{k=1}^{N} |\Delta_k|\cdot 1 = 1$$

For a Riemann integral to exist

$$\lim_{N\to\infty} \underline{\text{Sum}}_N = \lim_{N\to\infty} \overline{\text{sum}}_N .$$

$\therefore$ The Riemann integral does not exist in this case.

6.3

But here, we have

$$\lim_{N\to\infty} \overline{\text{Sum}}_N \neq \lim_{N\to\infty} \underline{\text{sum}}_N$$

$\therefore$ R.I. doesn't exist

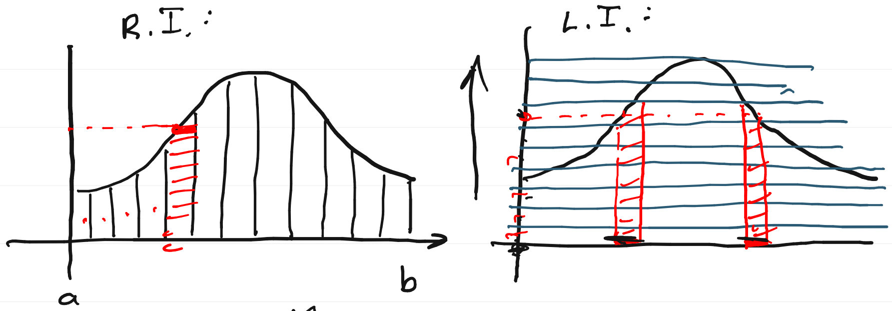
Yet by intuition

$$\underline{P(\mathbb{Q})} = \int_0^1 \mathbb{1}_{\mathbb{Q}}(r)\,dr = 0$$

We have a problem

This is why the Lebesgue was introduced.

6.4



1. $\int_{-\infty}^{\infty} 1_A(x)\, dx$ is defined for all $A \in B(\mathbb{R})$ if I use a Lebesgue integral

2. If the integrand has bounded absolute value, we can exchange the order of limits

6.5

How do we reconsile the difference between the R.I. and the L.I.?

Important Fact: If the Rieman Integral exists, the Lebesgue integral exists, and they are equal.

Proof: Measure Theory Course (MA 544)

This leads to the "Engineering compromise on integration"

Engineering Compromise on Integration

Given $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, for events $A \in \mathcal{B}(\mathbb{R})$ we calculate probabilities as

$$P(A) = \int_A f(r)\,dr = \int_{-\infty}^{\infty} f(r) \cdot 1_A(r)\,dr \quad \text{(L.I.)}$$

we interpret this as a Lebesgue integral, however for "friendly functions" (i.e., Riemann integrable functions) we compute the value of $P(A)$ using Riemann integration.

Compromise Approach is not Uncommon

	Theory	Practice
Computation:	Real	finite subset of the rationals.
Integration:	Lebesgue	Riemann

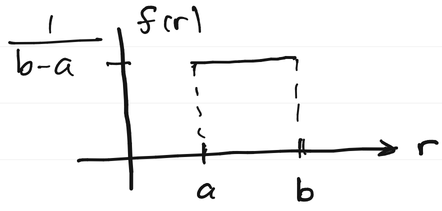
So we actually compute probabilities using the R.I. whenever possible.

We invoke the properties of the L.I. whenever it is beneficial to do so.

There are a number of useful pdfs we will want to know.

Ex. 1 The uniform pdf

$$f(r) = \frac{1}{b-a} \cdot 1_{[a,b]}(r) \quad , \quad r \in \mathbb{R}, \; a, b \in \mathbb{R}, \; a<b.$$



$$= \begin{cases} \frac{1}{b-a}, & r \in [a,b] \\ 0, & \text{elsewhere.} \end{cases}$$

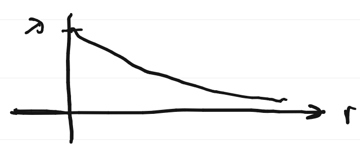
n.b. (i) $f(r) \geq 0$, $\forall r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\, dr = 1$.

Ex. 2 The exponential pdf

$$f(r) = \lambda e^{-\lambda r} \cdot 1_{[0,\infty)}(r) = \begin{cases} \lambda e^{-\lambda r}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

$\lambda > 0$



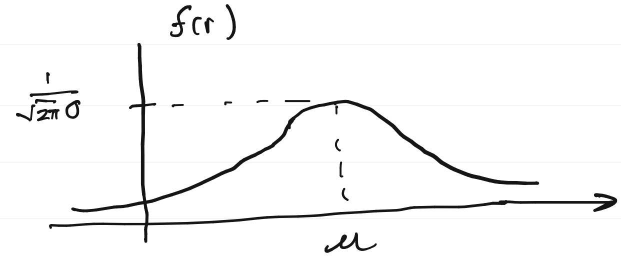
n.b. (i) $f(r) \geq 0$, $\forall r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\, dr = 1$.

Ex. 3 The Gaussian pdf

6.10

$$f(r) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(r-\mu)^2}{2\sigma^2} \right\} \qquad \begin{array}{l} r \in \mathbb{R} \\ \mu \in \mathbb{R} \\ \sigma > 0 \end{array}$$



(i) $f(r) \geq 0 \quad , r \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(r)\, dr = 1$

PMFs and PDFs You Should Know

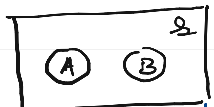
6.11

PMFs	PDFs
Binomial	Uniform
Geometric	Exponential
Poisson	Gaussian

Conditional Probability

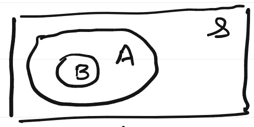
Given $(\mathcal{S}, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$, knowing that B has occured may tell us something about whether or not A has occured

Ex.1



If B occured, A did not

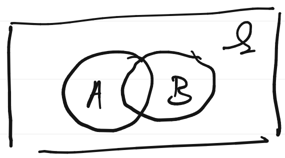
Ex.2



$B \subset A$

If B occured A occured,

Ex. 3



In general, knowing B has occured may change your belief that A has occured.

Defn: Given $(\mathcal{S}, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$, the conditional probability of A conditioned on B ("A given B") is

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)},$$

assuming $P(B) \neq 0$.

n.b. If $P(B) = 0$, then $P(A \cap B) = 0$, and this leaves us with $P(A|B) = \frac{0}{0}$ which is undefined.

n.b. 1. If $A \subset B$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \geq P(A)$$

2. If $B \subset A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3. If $A \cap B = \phi$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = \frac{0}{P(B)} = 0$$

4. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \geq P(A \cap B)$

$0 < P(B) \leq 1$

Fact: If $P(\cdot)$ (from $(S, \mathcal{F}, P)$) is a valid probability measure, then $P(\cdot | B)$ is also a valid probability measure for any $B \in \mathcal{F}$ such that $P(B) \neq 0$.

Proof: (exercise) Verify the axioms of probability hold for $P(\cdot | B)$.

$$(S, \mathcal{F}, P) \xRightarrow{\text{B has occurred}} (S, \mathcal{F}, P(\cdot | B))$$

n.b. $(S, \mathcal{F}, P(\cdot | B))$ is a valid prob. space because $(S, \mathcal{F}, P)$ is a valid prob. space.