EE600 Homework #2 Solutions Bell

1. Papoulis 2-2: $A = [2,5]$, $B = [3,6]$, $\mathcal{S} = \mathbb{R} = (-\infty, +\infty)$.

$$A \cup B = [2,5] \cup [3,6] = [2,6]$$

$$A \cap B = [2,5] \cap [3,6] = [3,5]$$

$$\begin{aligned}(A\cup B)\cap(\overline{A\cap B}) &= [2,6] \cap \overline{[3,5]} \\ &= [2,6] \cap ((-\infty,3)\cup(5,\infty)) \\ &= ([2,6]\cap(-\infty,3))\cup([2,6]\cap(5,\infty)) \\ &= [2,3) \cup (5,6]\end{aligned}$$

2. Papoulis 2-3: If $A \cap B = \phi \Rightarrow A$ and B are disjoint

Then $A \subset \overline{B} \Rightarrow P(A) \leq P(\overline{B})$.

n.b. If $A \subset \overline{B}$, then $\overline{B} = A \cup (\bar{A} \cap \overline{B})$ disjoint

$$\begin{aligned}\Rightarrow P(\overline{B}) &= P(A \cup (\bar{A}\cap\overline{B})) \\ &= P(A) + P(\bar{A}\cap\overline{B})\end{aligned}$$

and since $P(\bar{A}\cap\overline{B}) \geq 0 \Rightarrow P(A) \leq P(\overline{B})$.

3. Papoulis 2-4:

(a) $A = (A\cap B) \cup (A \cap \overline{B})$ and $B = (A\cap B)\cup(\bar{A}\cap B)$

disjoint disjoint

Now given that $P(A) = P(B) = P(A\cap B)$, it follows that

$$P(A) = P(A\cap B) + P(A\cap\overline{B}) \Rightarrow P(A\cap\overline{B}) = 0$$

$$P(B) = P(A\cap B) + P(\bar{A}\cap B) \Rightarrow P(\bar{A}\cap B) = 0$$

Now noting that $A\cap\overline{B}$ and $\bar{A}\cap B$ are disjoint, we have

$$\begin{aligned}P((A\cap\overline{B})\cup(\bar{A}\cap B)) &= P(A\cap\overline{B}) + P(\bar{A}\cap B) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

(3- continued)

(b) If $P(A) = P(B) = 1$, then it must also be the case that $P(A \cup B) = 1$, since

$$A \subset B \Rightarrow P(A) \leq P(A \cup B), \text{ but } P(A \cup B) \leq 1$$

So because

$$1 = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 2 - P(A \cap B)$$

$$\Rightarrow 1 = 2 - P(A \cap B) \Rightarrow P(A \cap B) = 1.$$

4. Papoulis 2-5: We know that for any two events E and F, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. Thus we have

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) \\
&\quad - P((A \cap B) \cup (A \cap C)) \\
&= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap C) - P(A \cap B) \\
&\quad - (-P(A \cap B \cap A \cap C)) \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
&\quad + P(A \cap B \cap C).
\end{aligned}$$

This can be generalized by induction to

$$\begin{aligned}
P(A_1 \cup A_2 \cup \cdots \cup A_n) &= P(A_1) + P(A_2) + \cdots + P(A_n) \\
&\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - \cdots - P(A_{n-1} \cap A_n) \\
&\quad + P(A_1 \cap A_2 \cap A_3) + \cdots + P(A_{n-2} \cap A_{n-1} \cap A_n) \\
&\quad \vdots \\
&\quad \pm P(A_1 \cap A_2 \cap \cdots \cap A_n).
\end{aligned}$$

5. Papoulis 2-6: There is not much to prove. Every event is an element of the event space, and hence can be written as a countable union of the elementary events $\{\zeta_1\}, \{\zeta_2\}, \dots, \{\zeta_n\}, \dots$. Thus all subsets of $\mathcal{S}$ are events contained in the event space.

6. Papoulis 2-7: We construct the smallest field by listing all unions, intersections, and complements of these elements and the elements that these operations generate. (Of course ϕ and $\mathcal{S}$ are in the field).

$\mathcal{S} = \{1,2,3,4\}$	$\{1,2,3\}$
ϕ	$\{2,3,4\}$
Given → $\{1\}$	$\{1,4\}$
Given → $\{2,3\}$	$\{4\}$

$$\therefore \quad \mathcal{F} = \sigma(\{\{1\},\{2,3\}\})$$
$$= \{\phi, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$$

7. Papoulis 2-8:

If $A \subset B$, $P(A) = 1/4$ and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$, because $A \cap B = A$ if $A \subset B$.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

8. Papoulis 2-9:

$$P(A|B\cap C)\,P(B|C) = \frac{P(A\cap B\cap C)}{P(B\cap C)} \cdot \frac{P(B\cap C)}{P(C)}$$

$$= \frac{P(A\cap B\cap C)}{P(C)} = P(A\cap B|C)$$

9. $B = (A\cap B)\cup(\bar{A}\cap B)$, and $(A\cap B)\cap(\bar{A}\cap B)=\phi$.
Thus $A\cap B$ and $\bar{A}\cap B$ are a partition of B.

So
$$P(B) = P((A\cap B)\cup(\bar{A}\cap B)) = P(A\cap B)+P(\bar{A}\cap B) \quad (*)$$

Furthermore, $P(\bar{A}) = 1-P(A)$. Thus we have

$$P(\bar{A}\cap B) - P(\bar{A})P(B) = \underbrace{P(B)-P(A\cap B)}_{P(\bar{A}\cap B)\text{ from }(*)} - \underbrace{[1-P(A)]}_{P(\bar{A})}P(B)$$

$$= P(B) - P(A\cap B) - P(B) + P(A)P(B)$$
$$= P(A)P(B) - P(A\cap B)$$

$$\therefore \quad P(A\cap B) - P(A)P(B) = P(\bar{A})P(B) - P(\bar{A}\cap B) \qquad (1)$$

Similarly, $P(A\cap\bar{B}) = P(A) - P(A\cap B)$ and $P(\bar{B}) = 1-P(B)$
Thus we have

$$P(A\cap\bar{B}) - P(A)P(\bar{B}) = P(A) - P(A\cap B) - [1-P(B)]P(A)$$
$$= P(A) - P(A\cap B) - P(A) + P(A)P(B)$$
$$= P(A)P(B) - P(A\cap B)$$

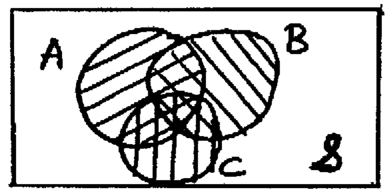
$$\therefore \quad P(A\cap B) - P(A)P(B) = P(A)P(\bar{B}) - P(A\cap\bar{B}) \qquad (2)$$

Thus from (1) and (2), we have the desired result:

$$P(A)P(B) - P(A\cap B) = P(\bar{A}\cap B) - P(\bar{A})P(B) = P(A\cap\bar{B}) - P(A)P(\bar{B}).$$

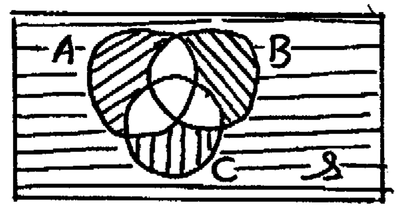
10. (a) At least one of the events A, B, C occurs

$\boxed{A \cup B \cup C}$



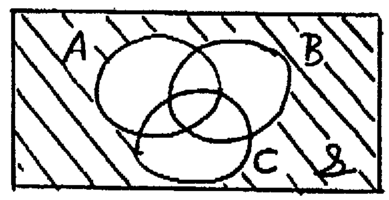
(b) At most one of the events A, B, C occurs.

$(\bar{A} \cap \bar{B} \cap \bar{C}) \cup (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$



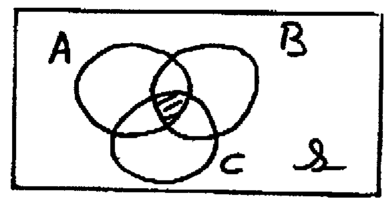
(c) None of the events A, B, C occurs

$\boxed{\overline{A \cup B \cup C} = \bar{A} \cap \bar{B} \cap \bar{C}}$



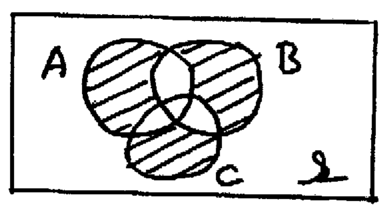
(d) All of the events occur

$\boxed{A \cap B \cap C}$



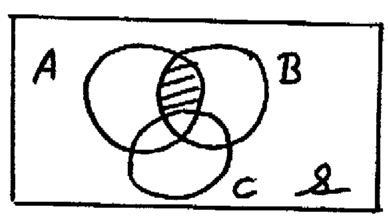
(e) Exactly one of the events A, B, C occurs

$\boxed{(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)}$



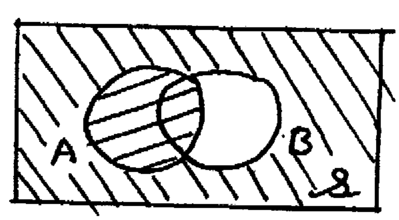
(f) A and B occur, but not C

$\boxed{A \cap B \cap \bar{C} = A \cap B - C}$



(g) A occurs, if not then B does not occur either.

$\boxed{A \cup (\bar{A} \cap \bar{B})}$



<u>11.</u> The sample space is

$$S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$$

(a) $A = \{(HHT), (HTH), (THH)\}$

$B = \{(HHT), (HTH), (THH), (HHH)\}$

$C = \{(THH), (THT), (TTH)\}$

(b) (i) $\overline{A} \cap B = \{(HHH)\}$: exactly three heads occur.

(ii) $\overline{A} \cap \overline{B} = \overline{A \cup B} = \overline{B}$ (because $A \subset B$) $=$

$= \{(HTT), (TTT), (TTH), (THT)\}$: at least two tails occur.

(iii) $A \cap C = \{(THH)\}$: a tails occurs on the first toss, and then the second and third tosses are both heads.