EE 600 Homework #4 Solutions Bell

1. Papoulis 4-1: Because $f(x)$ is an even ftn. $(f(-x)=f(x))$,

$$\int_{-\infty}^{x} f(\alpha)\,d\alpha = \int_{-x}^{\infty} f(\alpha)\,d\alpha \Rightarrow P(\{\mathbf{X}\le x\}) = P(\{\mathbf{X} > -x\})$$
$$= 1 - P(\{\mathbf{X}\le -x\})$$

$$\Longrightarrow F(x) = 1 - F(-x).$$

From the defn. of x_u, $u = F(x_u)$. It follows that $1-u = F(x_{1-u})$. Hence

$$1-u = 1-F(x_u) = F(-x_u) = F(x_{1-u}),$$

$$\Rightarrow -x_u = x_{1-u}.$$

2. Papoulis 4-2: From the symmetry of $f(\cdot)$ about η, we have

$$1-F(\eta+a) = F(\eta-a)$$

Hence $P(\{\eta-a<\mathbf{X}<\eta+a\}) = F(\eta+a)-F(\eta-a)$
$$= F(\eta+a) - [1-F(\eta+a)]$$
$$= 2F(\eta+a) - 1.$$

Thus we have

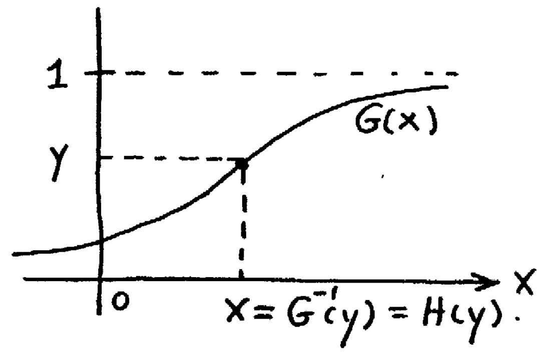
$$2F(\eta+a)-1 = 1-\alpha \Rightarrow F(\eta+a) = 1-\alpha/2$$

$$\Rightarrow \quad \eta+a = x_{1-\alpha/2}.$$

$$F(a-\eta) = \alpha/2 \Rightarrow a-\eta = x_{\alpha/2}.$$

3. Papoulis 4-11:

We wish to show that $F_X(x) = G(x)$ for $X(t_1) = H(t_1)$, where $H(\cdot) \triangleq G^{-1}(\cdot)$.



If $X(t_i) \le x$, then $t_i \le y = G(x)$, since $G(\cdot)$ is a monotonically increasing ftn. Thus

$$F_X(x) = P(\{X \le x\}) = P(\{t_i \le y\}) = y = G(x)$$

4. The result above shows that if you have a uniform random number generator with pdf $f_X(x) = 1_{(0,1)}(x)$ and hence cdf

$$F_X(x) = \begin{cases} 0 \ , & x \le 0 \\ x \ , & 0 < x \le 1 \\ 1 \ , & x > 1 \end{cases}$$

and you want to generate a R.V. with pdf $f_Z(z)$ and thus cdf $F_Z(z) = \int_{-\infty}^{z} f_Z(\alpha)\, d\alpha$, you can get such a R.V. Z as follows: Let $H(\cdot) = F_Z^{-1}(\cdot)$. Then operate on X with $H(\cdot)$ to get Z.

$$\boxed{\text{Uniform } (0,1)} \xrightarrow{X} \boxed{H(\cdot) = F_Z^{-1}(\cdot)} \longrightarrow Z \sim F_Z(z).$$

5. Papoulis 4-13: This is a repetition of 3 Bernouli trials, each having a prob. of success equal to ½. The R.V. X is equal to the number of successes in 3 trials.

$$P(\{X=0\}) = p_3(0) = \binom{3}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^3 = \frac{1}{8}$$

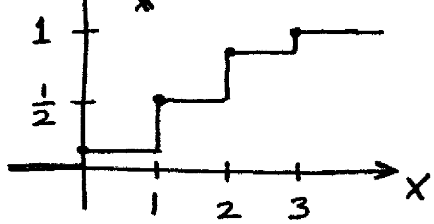
$$P(\{X=1\}) = p_3(1) = \binom{3}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^2 = \frac{3}{8}$$

$$P(\{X=2\}) = p_3(2) = \binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^1 = \frac{3}{8}$$

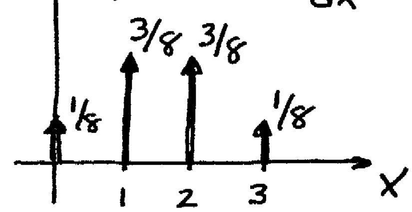
$$P(\{X=3\}) = p_3(3) = \binom{3}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^0 = \frac{1}{8}$$

Thus the cdf and pdf of the discrete RV X appears as follows:

$F_X(x) = P(\{X \le x\})$



$f_X(x) = \text{"}\dfrac{dF_X(x)}{dx}\text{"}$



6. Papoulis 4-16: If $X(\omega) \le Y(\omega)$, $\forall \omega \in S$, then $Y(\omega) \le \alpha \Rightarrow X(\omega) \le \alpha$ for all $\alpha \in \mathbb{R}$.

Thus $\{\omega : Y(\omega) \le \alpha\} \subset \{\omega : X(\omega) \le \alpha\}$, $\forall \alpha \in \mathbb{R}$

$\Rightarrow P(\{Y \le \alpha\}) \le P(\{X \le \alpha\})$, $\forall \alpha \in \mathbb{R}$

$\Rightarrow F_Y(\alpha) \le F_X(\alpha)$, $\forall \alpha \in \mathbb{R}$.

7. Papoulis 4-17: System goes into operation at time $t=0$. X = failure time. $\beta(t) = kt$ is the conditional failure rate, given by

$$\beta(t) = f_X(t \mid X > t).$$

Thus for small Δt, we have

$$P(\{t < X \le t+\Delta t\} \mid \{X > t\}) \simeq \beta(t)\Delta t$$

Thus

$$\beta(t)\Delta t \cong P(\{t < X \le t+\Delta t\} \mid \{X > t\})$$

$$= \frac{P(\{t < X \le t+\Delta t\} \cap \{X > t\})}{P(\{X > t\})}$$

$$= \frac{F_X(t+\Delta t) - F_X(t)}{1 - F_X(t)}$$

$$\Rightarrow \beta(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{F_X(t+\Delta t) - F_X(t)}{1 - F_X(t)} = \frac{f_X(t)}{1 - F_X(t)} = kt.$$

$$\Rightarrow \int_0^t \frac{f_X(\alpha)}{1 - F_X(\alpha)} d\alpha = \int_0^t k\alpha \, d\alpha = \frac{k\alpha^2}{2}\Big|_0^t = \frac{kt^2}{2}.$$

But $$\int_0^t \frac{f_X(\alpha)}{1 - F_X(\alpha)} d\alpha = -\int_1^{1-F_X(t)} \frac{du}{u} = -\ln[1 - F_X(t)] + \cancel{\ln 1}^{\,0}$$

$$\therefore -\ln[1 - F_X(t)] = \frac{kt^2}{2} \Rightarrow F_X(t) = \left[1 - e^{-kt^2/2}\right] 1_{[0,\infty)}(t).$$

Taking the derivative w.r.t. t to find the pdf, we get

$$f_X(t) = \frac{d}{dt} F_X(t) = kt e^{-kt^2/2} \cdot 1_{[0,\infty)}(t)$$

which is the Rayleigh density function.

8. Papoulis 4-19:

$$F_{\mathbb{X}}(x|A) = P(\{\mathbb{X} \le x\}|A) = \frac{P(\{\mathbb{X}\le x\}\cap A)}{P(A)}$$

$$= \frac{P(A|\{\mathbb{X}\le x\})\,P(\{\mathbb{X}\le x\})}{P(A)}$$

$$= \frac{P(A|\{\mathbb{X}\le x\})\,F_{\mathbb{X}}(x)}{P(A)}$$

9. Papoulis 4-21:

(a) The pdf of $\mathbb{P}$ is $f_{\mathbb{P}}(p) = 1_{(0,1)}(p) = \begin{cases} 1, & p\in(0,1) \\ 0, & \text{elsewhere} \end{cases}$

So $P(\{0.3 \le \mathbb{P} \le 0.7\}) = \int_{0.3}^{0.7} 1_{(0,1)}(p)\,dp = 0.4$

(b) Let $A = \{$6 Heads in 10 tosses$\}$

$$P(A|\mathbb{P}=p) = \binom{10}{6} p^6 (1-p)^{10-6}$$

$$f_{\mathbb{P}}(p|A) = \frac{P(A|\{\mathbb{P}=p\})\,f_{\mathbb{P}}(p)}{P(A)}$$

$$= \frac{P(A|\{\mathbb{P}=p\})\,f_{\mathbb{P}}(p)}{\int_0^1 P(A|\{\mathbb{P}=p\})\,f_{\mathbb{P}}(p)\,dp} = \frac{\binom{10}{6} p^6(1-p)^4\,1_{(0,1)}(p)}{\int_0^1 \binom{10}{6} p^6(1-p)^4\,dp}$$

$$= \frac{p^6(1-p)^4\,1_{(0,1)}(p)}{\int_0^1 p^6(1-p)^4\,dp} = \cdots = 2310\,p^6(1-p)^4\,1_{(0,1)}(p)$$

$$\therefore\ P(\{0.3 \le p \le 0.7\}|A) = \int_{0.3}^{0.7} 2310\,p^6(1-p)^4\,dp = 0.768.$$

10. To simplify the evaluation of

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} dx,$$

We make the substitution $\tilde{x} = x - \mu$ $(d\tilde{x} = dx)$. Then we have

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{\tilde{x}^2}{2\sigma^2} \right\} d\tilde{x} \qquad (*)$$

But since a closed-form antiderivative of the integrand is not known, we cannot evaluate $(*)$ directly. However we know that I is positive because the integrand is positive for all $\tilde{x} \in \mathbb{R}$. So if we can show $I^2 = 1$, then it follows that $I = 1$.

$$I^2 = I \cdot I = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-\tilde{x}^2}{2\sigma^2} \right) d\tilde{x} \right) \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-\tilde{y}^2}{2\sigma^2} \right) d\tilde{y} \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(\tilde{x}^2 + \tilde{y}^2)}{2\sigma^2} \right] d\tilde{x}\, d\tilde{y}, \quad \text{let } r = \sqrt{\tilde{x}^2 + \tilde{y}^2}, \quad \theta = \tan^{-1}(\tilde{y}/\tilde{x})$$

$$= \int_0^{2\pi} \int_0^{\infty} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2} \right) \left| \frac{\partial(\tilde{x}, \tilde{y})}{\partial(r, \theta)} \right| dr\, d\theta \qquad \tilde{x} = r\cos\theta, \quad \tilde{y} = r\sin\theta$$

$$\left| \frac{\partial(\tilde{x}, \tilde{y})}{\partial(r, \theta)} \right| = \left| \det \begin{pmatrix} \frac{\partial \tilde{x}}{\partial r} & \frac{\partial \tilde{y}}{\partial r} \\ \frac{\partial \tilde{x}}{\partial \theta} & \frac{\partial \tilde{y}}{\partial \theta} \end{pmatrix} \right| = |r| \left[\cos^2\theta + \sin^2\theta \right] = r$$

$$= \int_0^{2\pi} \int_0^{\infty} \frac{r}{2\pi\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2} \right) dr\, d\theta$$

$$= \int_0^{\infty} \frac{r}{\sigma^2} \exp\left(\frac{-r^2}{2\sigma^2} \right) dr = \exp\left(\frac{-r^2}{2\sigma^2} \right) \Bigg|_0^{\infty} = 1$$

$\therefore$ $I^2 = 1$, and because we know $I \geq 0 \Rightarrow I = 1$.

11. Let $A = \{\mu < X \le 2\mu\}$; $F_X(x) = \left[1 - e^{-x/\mu}\right] \cdot 1_{[0,\infty)}(x)$

$$F(x|A) = \frac{P(\{X \le x\} \cap A)}{P(A)} = \frac{P(\{X \le x\} \cap \{\mu < X \le 2\mu\})}{P(\{\mu < X \le 2\mu\})}$$

$$P(\{\mu < X \le 2\mu\}) = F_X(2\mu) - F_X(\mu) = e^{-1}(1 - e^{-1})$$

Note that

$$\{X \le x\} \cap \{\mu < X \le 2\mu\} = \begin{cases} \phi, & x \le \mu \\ (\mu, x], & \mu < x \le 2\mu \\ (\mu, 2\mu], & x > 2\mu \end{cases}$$

Thus

$$P(\{X \le x\} \cap \{\mu < X \le 2\mu\}) = \begin{cases} P(\phi) = 0, & x \le \mu \\ F_X(x) - F_X(\mu), & \mu < x \le 2\mu \\ F_X(2\mu) - F_X(\mu), & x > 2\mu \end{cases}$$

So

$$F(x|A) = \begin{cases} 0, & x \le \mu \\ \dfrac{1 - \exp\{-(x-\mu)/\mu\}}{1 - e^{-1}}, & \mu < x \le 2\mu \\ 1, & x > 2\mu \end{cases}$$

from which it follows that

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{\mu(1 - e^{-1})} e^{-(x-\mu)/\mu} \cdot 1_{(\mu, 2\mu]}(x)$$

$$= \begin{cases} \dfrac{1}{\mu(1 - e^{-1})} e^{-(x-\mu)/\mu}, & x \in (\mu, 2\mu] \\ 0, & \text{elsewhere.} \end{cases}$$