

14 SEMIDEFINITE PROGRAMMING IN SYSTEMS AND CONTROL THEORY

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14.1 INTRODUCTION

It has been long recognized that SDP constraints, i.e., Linear Matrix Inequalities (LMIs), arise naturally and frequently in the analysis of the solution of finite-dimensional differential equations that model control systems. The earliest LMI for systems and control is the “Lyapunov” LMI [495, p277]

$$P > 0, \quad A^T P + P A < 0. \quad (14.1.1)$$

This LMI is feasible if and only if every solution of

$$\frac{d}{dt}x(t) = Ax(t) \quad (14.1.2)$$

satisfies $\lim_{t \rightarrow \infty} x(t) = 0$. It turns out that a suitable P satisfying LMI (14.1.1) can be found simply by solving a Lyapunov equation, say $A^T P_0 + P_0 A + I = 0$. Then, (14.1.1) is feasible if and only if $P_0 > 0$.

Another important instance where LMIs arise in control theory is in absolute stability theory. In the 1940s, Lur’e, Postnikov, and others in the Soviet Union applied Lyapunov’s methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator [493]. Their stability criteria were expressed as LMIs. However, as numerical algorithms for checking the feasibility of these LMIs were unavailable then, the LMIs were reduced to polynomial inequalities which were then checked “by hand”. Connections between the LMIs that arise in absolute stability theory and certain frequency-domain inequalities were derived in the 1960s by Yakubovich, Popov, Kalman; these are known by various

names as the Kalman-Yakubovich-Popov (KYP) lemmas, or the positive- and bounded-real lemmas [29]. These connections enabled the graphical verification of the LMI conditions from absolute stability theory, and resulted in the celebrated Popov criterion, Circle criterion, Tsytkin criterion, and many variations.

In the 1960s and 1970s, the important role of LMIs in control theory was already recognized, especially in [819] and in [791]. Similar observations were explicitly made by several researchers, in [620] and [333], to name just a few. Thus, it can be said that by the mid-eighties, system and control theory were ripe for the application of SDP. Thus, with Nesterov and Nemirovskii's seminal work on interior point methods that apply directly to convex problems involving matrix inequalities [552, 562], there was a spurt in research efforts directed towards the numerical solution of systems and control problems using SDP; this has continued into this decade as well.

A number of publications can be found in the control literature that survey applications of SDP to the solution of system and control problems. Perhaps the most comprehensive list can be found in the book [128]. Since its publication, a number of papers have appeared chronicling further applications of SDP in control; we cite for instance the survey article [767], and the special issue of the International Journal of Robust and Nonlinear Control on *Linear Matrix Inequalities in Control Theory and Applications*, published in November-December, 1996. The growing popularity of LMI methods for control is also evidenced by the large number of publications at recent control conferences.

Our objective, in this chapter, is to describe the application of SDP towards the solution of problems from systems and control. All these applications fall under the topic of Robust Control, that of analysis of and design for control systems for which only inexact models are available. The list of references that we cite is by no means complete. In most cases, we have attempted to refer to the most relevant or up-to-date citation, which should serve as a starting point for a more careful literature search for interested readers.

14.2 CONTROL SYSTEM ANALYSIS AND DESIGN: AN INTRODUCTION

A number of control systems are well-modeled by finite-dimensional differential and/or difference equations. Systems modeled by differential equations are usually referred to as “continuous-time” systems, those modeled by difference equations are referred to as “discrete-time” systems, and those modeled by a mixture of differential and difference equations are called “hybrid systems”. For simplicity, we will henceforth focus on continuous-time systems, noting that all the problems we discuss in this chapter have a counterpart in discrete-time systems.

It is customary to represent the differential equations modeling continuous-time systems as a single first-order vector differential equation:

$$\frac{d}{dt}x(t) = f(x, w, u, t), \quad z(t) = g(x, w, u, t), \quad y(t) = h(x, w, u, t), \quad (14.2.3)$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_w}$, $u(t) \in \mathbf{R}^{n_u}$, $y(t) \in \mathbf{R}^{n_y}$ and $z(t) \in \mathbf{R}^{n_z}$. The function x is called the “state” of the system, while w and u are “inputs”, and z and y are “outputs”. w consists of exogenous inputs, i.e., inputs that we have no control over, such as noises, reference inputs etc. u consists of control inputs; we may set $u(t)$ to any value we wish, for every t . The outputs z are those of interest; these may consist, for instance, of components of x or even those of u . y consists of outputs that can be measured. f , g and h are either fixed functions, or are known only to satisfy some properties. The latter situation arises when the model only approximates the system; in this case, equations (14.2.3) are said to describe an “uncertain” system¹. (We will give specific examples shortly.)

Control system analysis problems consist of the study of the solutions of equations (14.2.3). Typical questions that arise in this context are “Are the solutions x of equations (14.2.3) bounded?” or “With $x(0) = 0$, how large can $\int_0^\infty z(t)^T z(t) dt$ be, over all w with $\int_0^\infty w(t)^T w(t) dt \leq 1$?” Control system design problems consist of designing control laws $u(t) = \mathcal{K}(y, t)$, so that with the control law in place, desired answers are obtained for the analysis questions. Figure 14.1 shows a block diagram of the control system model (14.2.3) with the controller, i.e., the control law, in place. In this chapter, we present some examples of the application of SDP towards solving analysis and design problems in uncertain control systems.

14.2.1 Linear fractional representation of uncertain systems

We now focus on a special instance of system (14.2.3), consisting of an interconnection of a linear time-invariant system and an “uncertainty” or “perturbation” in the feedback loop. This model has found wide applicability in the analysis and design of control systems for which only imperfect models are

¹Control system models must often explicitly incorporate in them “uncertainties”, which model a number of factors, including: dynamics that are neglected to make the model tractable, as with large scale structures; nonlinearities that are either hard to model or too complicated; and parameters that are not known exactly, either because they are hard to measure or because of varying manufacturing conditions. Robust control deals with the analysis of and design for such control system models.

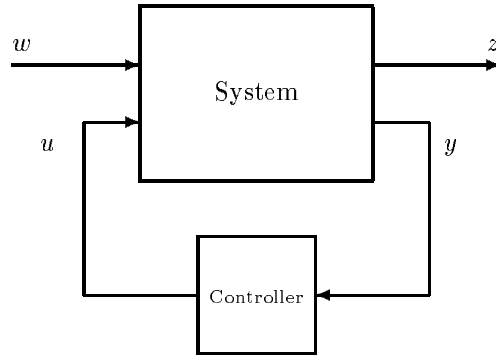


Figure 14.1 A standard controller design framework.

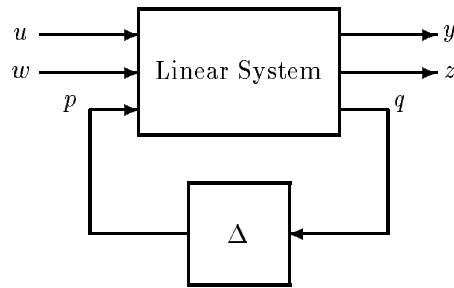


Figure 14.2 A common framework for robustness analysis and robust synthesis.

available; see for example, [280]. The model is described by

$$\begin{aligned}
 \frac{d}{dt}x(t) &= Ax(t) + B_p p(t) + B_u u(t) + B_w w(t), \\
 q(t) &= C_q x(t) + D_{qp} p(t) + D_{qu} u(t) + D_{qw} w(t), \\
 y(t) &= C_y x(t) + D_{yp} p(t) + D_{yu} u(t) + D_{yw} w(t), \\
 z(t) &= C_z x(t) + D_{zp} p(t) + D_{zu} u(t) + D_{zw} w(t), \\
 p(t) &= \Delta(q, t),
 \end{aligned}
 \tag{14.2.4}$$

where $p \in \mathbf{R}^m$, $q \in \mathbf{R}^m$, A , B_p , B_u , B_w , C_q , C_y , C_z , D_{yp} , D_{yu} , D_{yw} , D_{qp} , D_{qu} , D_{qw} , D_{zp} , D_{zu} and D_{zw} are real matrices of appropriate sizes. $\Delta : \mathbf{L}_2^m[0, \infty) \rightarrow \mathbf{L}_2^m[0, \infty)$ is in general a nonlinear operator representing the “uncertainty” in modeling, and is known or assumed to lie in some set $\mathbf{\Delta}$. Often $\mathbf{\Delta}$ contains the origin, i.e., $\Delta = 0$; the linear time-invariant system that results is called the “nominal model”. A block diagram of this system model is shown in Figure 14.2.

Many commonly encountered systems with structured and/or parametric uncertainties can be represented by the system model (14.2.4) [181, 369]. In the control literature, model (14.2.4) is also known as the “Linear Fractional Representation” of the uncertain system, or simply an LFR system. Usually, additional information about the size of the uncertainty (typically some bound on the norm of $\Delta \in \mathbf{\Delta}$), its structure (i.e., diagonal or block-diagonal), and nature (for instance, sector-bounded memoryless, linear time-invariant (LTI) or parametric, etc) is available.

14.2.2 Polytopic systems

Polytopic systems form a special class of LFR systems. For these systems, there exists an extensive body of work on analysis and synthesis using quadratic Lyapunov functions and SDP [128]. These systems are described by

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t)x(t) + B_u(t)u(t) + B_w(t)w(t), \\ y(t) &= C_y(t)x(t) + D_{yu}(t)u(t) + D_{yw}(t)w(t), \\ z(t) &= C_z(t)x(t) + D_{zu}(t)u(t) + D_{zw}(t)w(t), \end{aligned} \quad (14.2.5)$$

$$\Sigma(t) = \begin{bmatrix} A(t) & B_u(t) & B_w(t) \\ C_y(t) & D_{yu}(t) & D_{yw}(t) \\ C_z(t) & D_{zu}(t) & D_{zw}(t) \end{bmatrix} \in \Xi$$

where

$$\Xi = \text{Co} \left\{ \begin{bmatrix} A_1 & B_{u,1} & B_{w,1} \\ C_{y,1} & D_{yu,1} & D_{yw,1} \\ C_{z,1} & D_{zu,1} & D_{zw,1} \end{bmatrix}, \dots, \begin{bmatrix} A_L & B_{u,L} & B_{w,L} \\ C_{y,L} & D_{yu,L} & D_{yw,L} \\ C_{z,L} & D_{zu,L} & D_{zw,L} \end{bmatrix} \right\}, \quad (14.2.6)$$

where **Co** denotes the convex hull. (The matrices $\begin{bmatrix} A_i & B_{u,i} & B_{w,i} \\ C_{y,i} & D_{yu,i} & D_{yw,i} \\ C_{z,i} & D_{zu,i} & D_{zw,i} \end{bmatrix}$, $i = 1, \dots, L$ are given.)

14.2.3 Robust stability analysis and design problems

We consider questions of stability analysis and stabilizing controller synthesis for both polytopic systems (14.2.5) and the more general LFR systems (14.2.4):

- (P1) With w and u identically zero, does the state x of system (14.2.5) (respectively system (14.2.4)) satisfy $\lim_{t \rightarrow \infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that the system (14.2.5) (respectively system (14.2.4)) is “robustly stable over Ξ (respectively $\mathbf{\Delta}$)”.
- (P2) With w identically zero, does there exist a control law u such that the state x of system (14.2.5) (respectively system (14.2.4)) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that the system (14.2.5) (respectively system (14.2.4)) is “robustly stabilizable over Ξ (respectively $\mathbf{\Delta}$)”.

Each of these “robust stability” questions has a “robust performance” counterpart: For a robustly stable system, measures of performance—usually with smaller values being better—can be defined that quantify how good the map from the exogenous inputs w to the outputs of interest z is. Robust performance analysis questions then ask how large these performance measures can be over Ξ or Δ . Robust performance design questions concern the design of control laws that minimize the largest values of the performance measures over Ξ or Δ . However, for simplicity, we will not consider robust performance problems further in the sequel.

One approach towards answering question (P1) uses the notion of *quadratic stability*. A system is said to be quadratically stable if there exists a single positive-definite quadratic Lyapunov function $V(\zeta) = \zeta^T P \zeta$ that decreases along every trajectory of the system. For system (14.2.5), a sufficient and necessary condition for quadratic stability can be directly formulated in terms of a finite number of LMIs [128]. For system (14.2.4), in general, only sufficient conditions for quadratic stability are known; these are stated in terms of a finite number of LMIs.

A system can be robustly stable without being quadratically stable, and more general Lyapunov functions can be employed to derive weaker sufficient conditions for robust stability. For instance, when the state-space matrices of the polytopic system (14.2.5) vary slowly with time, stability analysis using parameter-dependent Lyapunov functions usually leads to less conservative robust stability conditions than the analysis based on quadratic Lyapunov functions [246]. For the LFR system (14.2.4), the framework of integral quadratic constraints (IQCs) [509] provides a systematic method for deriving sufficient conditions for robust stability that are weaker than quadratic stability. In many cases, this framework can be interpreted as searching for more general Lyapunov functions.

In addressing the problem of controller synthesis (P2), there are several possibilities for generating the control input $u(t)$. Perhaps the simplest control law is that of constant state-feedback, $u(t) = Kx(t)$, where K is a real matrix. Of course, in order to implement a state-feedback scheme, the state $x(t)$ has to be measurable at every time t . If only the measured output y is available for generating u , output feedback control laws of the form $u = \mathcal{K}(y, t)$ can be envisioned; a simple example of such a control law is constant output feedback $u = Ky(t)$. If in addition to the measured output, the uncertainty Σ in a polytopic system (or Δ in an LFR system) is measurable in real time [578, 44], a control law $u = \mathcal{K}(y, \Sigma, t)$ (or $u = \mathcal{K}(y, \Delta, t)$) that explicitly depends on the uncertainty can be implemented. This is the so-called *gain-scheduled controller*².

²The use of the term “gain-scheduled” in the context of this chapter refers to the framework of LMI-based gain-scheduling techniques [73, 578, 245, 812, 44, 145, 194, 659, 42]. This is to be contrasted from the classical gain-scheduling controller synthesis where several controllers are designed for the system under different operating conditions, with the actual control law

The problem of synthesizing robustly stabilizing constant state-feedback for both polytopic and LFR systems can be formulated as LMI feasibility problems [128]. However, no convex reformulation is known for the problem of even constant output feedback synthesis for even polytopic systems. However, gain-scheduled controllers appear to hold promise: Designing gain-scheduled output feedback controllers for polytopic systems using quadratic Lyapunov functions can be reduced to the solution of an optimization problem with a finite number of LMIs [73, 44, 42]. For LFR systems, conditions for the existence of robustly stabilizing gain-scheduled output feedback controllers, derived using quadratic Lyapunov functions, result in a finite number of LMIs [194, 73, 43, 660]. As for the stability criteria derived using parameter-dependent Lyapunov functions or in the IQC framework, although they may yield less conservative conditions for robust stability, the corresponding conditions for the existence of robustly stabilizing controllers (gain-scheduled or otherwise) turn out to be nonconvex.

We next present, in detail, some of the SDP-based robust stability analysis and controller synthesis techniques that we have summarized so far. The specific problems that we consider are:

- For polytopic systems (14.2.5), robust stability analysis, state-feedback synthesis, and gain-scheduled controller synthesis, using quadratic Lyapunov functions.
- For LFR systems (14.2.4), robust stability analysis in the IQC framework.
- For a class of LFR systems (14.2.4), state-feedback synthesis and gain-scheduled controller synthesis, using quadratic Lyapunov functions.

14.3 ROBUSTNESS ANALYSIS AND DESIGN FOR LINEAR POLYTOPIC SYSTEMS USING QUADRATIC LYAPUNOV FUNCTIONS

For the polytopic system (14.2.5), we derive a necessary and sufficient condition for quadratic stability. This stability condition in turn is used to design state feedback and gain-scheduled output feedback controllers that stabilize system (14.2.5). Conditions for the existence of these controllers are formulated in terms of a finite number of LMIs.

14.3.1 Robust stability analysis

Setting u and w to be identically zero in equations (14.2.5) yields the following state equation

$$\frac{dx}{dt} = A(t)x(t), \quad A(t) \in \mathbf{Co} \{A_1, \dots, A_L\}. \quad (14.3.7)$$

“switching” between the locally designed controllers using some “scheduling” scheme [668, 669].

This autonomous system is quadratically stable if there exists a quadratic Lyapunov function $V(\psi) = \psi^T P \psi$, with $P > 0$ such that $dV(x(t))/dt < 0$ for every nonzero solution x . Since

$$\frac{d}{dt}V(x(t)) = x(t)^T (A(t)^T P + PA(t)) x(t), \quad (14.3.8)$$

system (14.3.7) is quadratically stable if and only if there exists P such that

$$P = P^T > 0, \quad A(t)^T P + PA(t) < 0, \quad A(t) \in \mathbf{Co}\{A_1, \dots, A_L\}, \quad (14.3.9)$$

or equivalently

$$P = P^T > 0, \quad A_i^T P + PA_i < 0, \quad i = 1, \dots, L. \quad (14.3.10)$$

V is sometimes called a “simultaneous quadratic Lyapunov function” since it proves the stability of every element of $\mathbf{Co}\{A_1, \dots, A_L\}$. Thus, determining quadratic stability is an SDP feasibility problem.

14.3.2 Stabilizing state-feedback controller synthesis

Consider the system (14.3.7) with a control input u that is generated via state-feedback:

$$\frac{d}{dt}x(t) = A(t)x(t) + B_u(t)u(t), \quad u(t) = Kx(t), \quad (14.3.11)$$

where

$$[A(t) \ B_u(t)] \in \mathbf{Co}\{[A_1 \ B_{u,1}], \dots, [A_L \ B_{u,L}]\}. \quad (14.3.12)$$

Our objective is to design the matrix K such that (14.3.11) is quadratically stable. This is a “quadratic stabilizability” problem.

System (14.3.11) is quadratically stable with some state-feedback gain K if there exist P and K such that

$$P = P^T > 0, \quad (A_i + B_{u,i}K)^T P + P(A_i + B_{u,i}K) < 0, \quad i = 1, \dots, L. \quad (14.3.13)$$

Note that the matrix inequality (14.3.13) is *not* jointly convex in P and K . However, with the bijective transformation $Q \triangleq P^{-1}$, $Y \triangleq KP^{-1}$, we may rewrite (14.3.13) as

$$Q = Q^T > 0, \quad (A_i + B_{u,i}YQ^{-1})^T Q^{-1} + Q^{-1}(A_i + B_{u,i}YQ^{-1}) < 0, \quad i = 1, \dots, L. \quad (14.3.14)$$

Multiplying the second inequality on the left and right by Q (such a congruence preserves the inequality) we get an LMI in Q and Y :

$$Q = Q^T > 0, \quad QA_i^T + Y^T B_{u,i}^T + A_i Q + B_{u,i} Y < 0, \quad i = 1, \dots, L. \quad (14.3.15)$$

If this LMI problem is feasible and Q and Y are feasible solutions, then the Lyapunov function $V(\psi) = \psi^T Q^{-1} \psi$ proves quadratic stability of the closed-loop system, with state-feedback $u(t) = YQ^{-1}x(t)$. In other words, we can synthesize a constant state-feedback controller that (quadratically) stabilizes the polytopic system (14.3.7) by solving an LMI feasibility problem.

14.3.3 Gain-scheduled output feedback controller synthesis

Consider the system (14.3.7) with a control input u and a measured output y :

$$\frac{d}{dt}x(t) = A(t)x(t) + B_u u(t), \quad y(t) = C_y x(t), \quad (14.3.16)$$

where $A(t) \in \mathbf{Co}\{A_1, \dots, A_L\}$. (For simplicity, we have set $D_{yu} = 0$. It turns out that this assumption entails no loss of generality.)

When attempting to synthesize a simple constant output feedback control $u(t) = Ky(t)$, one arrives at the following condition for quadratic stability of the closed-loop system:

$$P = P^T > 0, \quad (A(t) + B_u K C_y)^T P + P(A(t) + B_u K C_y) < 0, \quad (14.3.17)$$

and there is no way known to derive equivalent LMI conditions (unlike with state-feedback). However, in many cases, although the uncertainties are not known a priori, they can be measured in real time, so that the control law can be allowed to “schedule” itself according to the measured uncertainties. Then, it turns out that conditions for the existence of so-called gain-scheduled control laws that stabilize the system as well as expressions for the control laws themselves can be derived using SDP. We therefore first pose the gain-scheduled controller design problem below, and describe its solution.

Consider system (14.3.16), where

$$A(t) = \sum_{i=1}^L \theta_i(t) A_i, \quad \sum_{i=1}^L \theta_i(t) = 1.$$

$\theta(t) = [\theta_1(t), \dots, \theta_L(t)]$ is unknown a priori, but can be measured in real time. We will design a gain-scheduled output feedback controller

$$\frac{d}{dt}x_k(t) = A_k(\theta(t))x_k(t) + B_k(\theta(t))y(t), \quad u(t) = C_k(\theta(t))x_k(t) + D_k(\theta(t))y(t),$$

where $x_k(t) \in \mathbf{R}^{n_k}$, and $A_k(\cdot)$, $B_k(\cdot)$, $C_k(\cdot)$ and $D_k(\cdot)$ are real valued functions of $\theta(t)$.

The state space representation of the closed loop system is

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} = A_{\text{cl}}(t) \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix},$$

where

$$\begin{aligned} A_{\text{cl}}(t) &= A_0(t) + \mathbf{B}\Omega(\theta(t))\mathbf{C}, \\ \Omega(\theta(t)) &= \begin{bmatrix} A_k(\theta(t)) & B_k(\theta(t)) \\ C_k(\theta(t)) & D_k(\theta(t)) \end{bmatrix}, \quad A_0(t) = \begin{bmatrix} A(t) & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & I \\ C_y & 0 \end{bmatrix}. \end{aligned} \quad (14.3.18)$$

Now, the closed loop system is quadratically stable if there exists $P = P^T > 0$ such that

$$A_{\text{cl}}(t)^T P + P A_{\text{cl}}(t) < 0 \quad (14.3.19)$$

for all possible values of $\theta(t)$.

The following two lemmas play a central role in the derivation of LMI conditions that are equivalent to (14.3.19). The first lemma is called the Elimination lemma (see for example, [128]). The second lemma is called the Completion lemma [578].

Lemma 14.3.1 *Given matrices $G \in \mathbf{R}^{n \times n}$, $U \in \mathbf{R}^{n \times p}$, and $V \in \mathbf{R}^{n \times q}$, there exists $\Omega \in \mathbf{R}^{p \times q}$ such that*

$$G + U\Omega V^T + V\Omega^T U^T > 0$$

if and only if

$$U_{\perp}^T G U_{\perp} > 0 \quad \text{and} \quad V_{\perp}^T G V_{\perp} > 0,$$

where U_{\perp} and V_{\perp} are the orthogonal complements of U and V respectively.

Lemma 14.3.2 *Let $X = X^T \in \mathbf{R}^{n \times n}$ and $Y = Y^T \in \mathbf{R}^{n \times n}$ be positive-definite matrices. There exist $X_2 \in \mathbf{R}^{n \times r}$, $X_3 \in \mathbf{R}^{r \times r}$, $Y_2 \in \mathbf{R}^{n \times r}$ and $Y_3 \in \mathbf{R}^{r \times r}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \quad \text{and} \quad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r,$$

where I_n denotes an $n \times n$ identity matrix.

Using the Elimination Lemma 14.3.1 and Completion Lemma 14.3.2 and some straightforward matrix manipulations, it can be shown that there exists a full order stabilizing gain-scheduled output feedback controller, if there exist $R = R^T \in \mathbf{R}^{n \times n}$ and $S = S^T \in \mathbf{R}^{n \times n}$ such that the following matrix inequalities hold:

$$\begin{aligned} N_R^T (A(t)R + RA(t)^T) N_R &< 0, & \begin{bmatrix} S & I \\ I & R \end{bmatrix} &\geq 0, \\ N_S^T (A(t)^T S + SA(t)) N_S &< 0, & \end{aligned} \quad (14.3.20)$$

where N_R and N_S are matrices whose columns comprise the bases of the null spaces of B_u^T and C_y respectively. Since $A(t)$ lies in a polytope with vertices A_1, \dots, A_L , condition (14.3.20) is equivalent to

$$\begin{aligned} \begin{bmatrix} S & I \\ I & R \end{bmatrix} &\geq 0, & N_R^T (A_i R + RA_i^T) N_R &< 0, \\ & & N_S^T (A_i^T S + SA_i) N_S &< 0, \quad i = 1, \dots, L. \end{aligned} \quad (14.3.21)$$

Thus, we have an LMI condition that is necessary and sufficient for the existence of a quadratically stabilizing gain-scheduled output feedback controller for the polytopic system (14.3.16). We next describe an algorithm for explicitly constructing a family of gain-scheduled output feedback controllers that are guaranteed to stabilize the system.

Step 1. Design controllers at each vertex of the polytope $\mathbf{Co}\{A_1, \dots, A_L\}$.

Let (R, S) be a feasible solution to (14.3.21). With $S > R^{-1}$, define $P_{12} = (S - R^{-1})^{1/2}$, $Q_{12} = -RP_{12}$ and

$$P = \begin{bmatrix} S & P_{12} \\ P_{12}^T & I \end{bmatrix}.$$

It is easy to check that $P > 0$ and

$$P^{-1} = \begin{bmatrix} R & Q_{12} \\ Q_{12}^T & I - P_{12}^T Q_{12} \end{bmatrix} > 0.$$

Let $A(\theta(t)) = A_i$, $i = 1, \dots, L$. With P defined above, LMI (14.3.19) is feasible (from the feasibility of (14.3.21)). Solve (14.3.19) for Ω_i , which comprises the state space matrices of the controller corresponding to the vertex A_i .

Step 2. Design the gain-scheduled controller.

For any $A(t) \in \mathbf{Co}\{A_1, \dots, A_L\}$, solve the set of linear equations

$$A(t) = \sum_{i=1}^L \theta_i(t) A_i, \quad \sum_{i=1}^L \theta_i(t) = 1,$$

to get $\theta_i(t)$. Inequality (14.3.19) is affine on $\Omega(\theta(t))$. Therefore,

$$\Omega(\theta(t)) = \sum_{i=1}^L \theta_i(t) \Omega_i, \quad (14.3.22)$$

comprises the state space matrices of a stabilizing gain-scheduled output feedback controller.

14.4 ROBUST STABILITY ANALYSIS OF LFR SYSTEMS IN THE IQC FRAMEWORK

We next focus on the robust stability analysis of system (14.2.4). With the inputs u and w identically zero, the autonomous uncertain system is described by

$$\frac{d}{dt} x(t) = Ax(t) + B_p p(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad p = \Delta(q, t). \quad (14.4.23)$$

Let $H(s) = C_q(sI - A)^{-1}B_p + D_{qp}$.

Recall that the uncertainty Δ is known to lie in some set $\mathbf{\Delta}$. This information can be represented in an elegant and mathematically tractable way using the framework of Integral Quadratic Constraints³ or IQCs [509].

We first provide a brief description of the IQC framework (for notation, terminology and details, we refer the reader to [509]). Two signals $p \in \mathbf{L}_2^m[0, \infty)$ and $q \in \mathbf{L}_2^m[0, \infty)$, with Fourier Transforms \hat{p} and \hat{q} respectively (assuming that the Fourier Transforms exist), are said to “satisfy the IQC defined by Π ”, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{p}(j\omega) \\ \hat{q}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{p}(j\omega) \\ \hat{q}(j\omega) \end{bmatrix} d\omega \geq 0, \quad (14.4.24)$$

where $\Pi : j\mathbf{R} \rightarrow \mathbf{C}^{2m \times 2m}$ is a measurable Hermitian function, bounded on the imaginary axis. We also say that $\Delta : \mathbf{L}_2^m[0, \infty) \rightarrow \mathbf{L}_2^m[0, \infty)$ “satisfies the IQC Π ”, if for every $q \in \mathbf{L}_2^m[0, \infty)$, q and Δq satisfy the IQC defined by Π . With this terminology, we assume that $\mathbf{\Delta}$ lies in the set

$$\mathbf{\Delta} = \left\{ \Delta \mid \begin{array}{l} \text{For every } \Pi \in \mathbf{\Pi}, \text{ for every } \tau \in [0, 1], \\ \tau\Delta \text{ satisfies the IQC defined by } \Pi \end{array} \right\},$$

where $\mathbf{\Pi}$ is some specified set that can be thought of as summarizing the information known about Δ . In all the examples that we will consider shortly, the elements of $\mathbf{\Pi}$ have the following property:

$$\text{Partitioning any } \Pi \in \mathbf{\Pi} \text{ as } \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}, \quad (14.4.25)$$

for some $\epsilon > 0$, for all $\omega \in \mathbf{R}$, $\Pi_{11}(j\omega) \geq 2\epsilon$ and $\Pi_{22}(j\omega) \leq -2\epsilon$.

We first review a sufficient condition [509, Theorem 1] for the robust stability of system (14.4.23) over $\mathbf{\Delta}$.

Theorem 14.4.1 *Suppose that the system represented by the equations*

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad p(t) = \tau \Delta(q, t),$$

³The framework of integral quadratic constraints helps unify a number of sufficient conditions for the robust stability of system (14.4.23) over $\mathbf{\Delta}$. When Δ can be *any* operator satisfying an \mathbf{L}_2 -gain bound, the small-gain theorem provides a necessary and sufficient condition for robust stability [173]. When Δ is structured—say diagonal—the small gain condition is no longer necessary for stability; diagonal scalings can then be used to derive less conservative robust stability conditions [181, 654]. In addition, if Δ is a memoryless time-invariant sector-bounded nonlinearity, the celebrated Popov criterion yields a sufficient condition for robust stability (see for example, [173]). When Δ is LTI or parametric, the well-known μ analysis and K_m analysis methods provide sufficient conditions for robust stability [58, 144, 196]. Besides enabling the rederivation and theoretical analysis of all these sufficient conditions, the IQC framework also lends itself to the derivation of other, new, sufficient conditions for robust stability [509].

is well-posed for any $\tau \in [0, 1]$ and any $\Delta \in \mathbf{\Delta}$. Then, if there exist $\Pi \in \mathbf{\Pi}$ and $\epsilon > 0$ such that

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \leq -2\epsilon I, \text{ for all } \omega \in \mathbf{R}, \quad (14.4.26)$$

then system (14.4.23) is robustly stable over $\mathbf{\Delta}$.

In general, $\mathbf{\Pi}$ —the set defining the IQCs corresponding to $\mathbf{\Delta}$ —is not described by a finite number of variables. In order to reduce the number of optimization variables to a finite number, a subset of $\mathbf{\Pi}$ is defined as

$$\mathbf{\Pi}_{\text{fin}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}; \Pi_{ij} = W(j\omega)^* T_{ij} W(j\omega); \\ W(j\omega) = \begin{bmatrix} C_W(j\omega I - A_W)^{-1} B_W \\ D_W \end{bmatrix}; \\ \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & -T_{22} \end{bmatrix} \in \mathbf{T}; \text{ for some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, \\ W(j\omega)^* T_{11} W(j\omega) \geq 2\epsilon I, \quad W(j\omega)^* T_{22} W(j\omega) \geq 2\epsilon I. \end{array} \right. \quad (14.4.27)$$

where $A_W \in \mathbf{R}^{n_w \times n_w}$, $B_W \in \mathbf{R}^{n_w \times m}$, $C_W \in \mathbf{R}^{N_1 \times n_w}$, $D_W \in \mathbf{R}^{N_2 \times m}$, and \mathbf{T} is an appropriately chosen subspace of $\mathbf{R}^{2(N_1+N_2) \times 2(N_1+N_2)}$, (We will give specific examples later.)

The “frequency-domain” condition (14.4.26) plays a key role in the robust stability analysis. The celebrated positive-real lemma⁴(or Kalman-Yakubovich-Popov lemma), which provides a connection between frequency-domain conditions on transfer functions and the underlying state-space matrices, enables us to derive an LMI that is equivalent to condition (14.4.26).

Lemma 14.4.1 (Positive Real Lemma [628]) *Let $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $M = M^T \in \mathbf{R}^{(m+n) \times (m+n)}$, with A having no eigenvalues on the imaginary axis. Then, the following statements are equivalent.*

1. For some $\epsilon > 0$,

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \geq 2\epsilon I, \quad \text{for all } \omega \in \mathbf{R}.$$

2. There exists a symmetric matrix $P = P^T$ such that

$$\begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} < M.$$

⁴A number of different versions of the positive-real lemma can be found in the literature; see for example [815, 816, 372, 373, 29, 128, 628].

Using this lemma, checking if condition (14.4.26) holds can be reduced to an LMI.

Theorem 14.4.2 *Let*

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A_W & B_W C_q & 0 \\ 0 & A & 0 \\ 0 & 0 & A_W \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C_q & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} B_W D_{qp} \\ B_p \\ B_W \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 \\ D_{qp} \\ 0 \\ I \end{bmatrix}, \quad E = \begin{bmatrix} C_W & 0 \\ 0 & D_W \end{bmatrix}. \end{aligned}$$

Then, condition (14.4.26) holds with $\Pi(j\omega) \in \mathbf{\Pi}_{\text{fin}}$, if and only if the LMIs

$$\begin{aligned} M_1 &= E^T T_{11} E - \begin{bmatrix} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{bmatrix} > 0, \\ M_2 &= E^T T_{22} E - \begin{bmatrix} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{bmatrix} > 0, \\ \begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} M_1 & E^T T_{12} E \\ E^T T_{12}^T E & -M_2 \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0, \\ P = P^T, \quad Q_1 = Q_1^T, \quad Q_2 = Q_2^T, \quad \begin{bmatrix} T_{11} & T_{12} \\ T_{12}^T & -T_{22} \end{bmatrix} \in \mathbf{T} \end{aligned}$$

are feasible.

We now describe the results that can be obtained from an application of Theorem 14.4.2 in several commonly-encountered situations.

14.4.1 Diagonal nonlinearities

Consider the special case when Δ is a “diagonal” uncertainty, i.e., if $p(t) = \Delta(q, t)$, then $p_i(t) = \delta_i(q_i, t)$, or the i th component of p is purely a function of the i th component of q . Moreover, suppose that the \mathbf{L}_2 gain of Δ does not exceed one, i.e., if $p(t) = \Delta(q, t)$, then

$$\int_0^T p(t)^T p(t) dt \leq \int_0^T q(t)^T q(t) dt, \quad \text{for all } T > 0. \quad (14.4.28)$$

Then, it turns out that Δ satisfies every IQC from the set

$$\mathbf{\Pi}^{\text{DNL}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix} \text{ for all } \omega \in \mathbf{R}, \\ W \in \mathbf{R}^{m \times m}, \quad W > 0 \text{ and diagonal} \end{array} \right. \right\}. \quad (14.4.29)$$

Note that $\mathbf{\Pi}^{\text{DNL}}$ is already described by a finite number of variables so that $\mathbf{\Pi}^{\text{DNL}} = \mathbf{\Pi}_{\text{fin}}^{\text{DNL}}$ and is defined by (14.4.27), where A_W , B_W and C_W are vacuous, $D_W = I$, and $\mathbf{T} = \mathbf{\Pi}^{\text{DNL}}$.

From Theorem 14.4.2, condition (14.4.26) is equivalent to the LMI

$$P = P^T, W \in \mathbf{R}^{m \times m} \text{ and diagonal, } W > 0,$$

$$\begin{bmatrix} A^T P + PA & PB_p \\ B_p^T P & 0 \end{bmatrix} + \begin{bmatrix} C_q^T & 0 \\ D_{qp}^T & I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix} \begin{bmatrix} C_q & D_{qp} \\ 0 & I \end{bmatrix} < 0. \quad (14.4.30)$$

It turns out that this LMI is equivalent to the condition that a diagonally scaled \mathbf{H}_∞ norm of $H(s) = C_q(sI - A)^{-1}B_p + D_{qp}$ is less than one; see [188, 128].

14.4.2 Parametric uncertainties

Suppose Δ is a constant real matrix with a specified block-diagonal structure, and with a spectral norm that does not exceed one:

$$\Delta = \text{diag}(D_1, \dots, D_M, d_1 I_{\ell_1}, \dots, d_N I_{\ell_N}),$$

$D_i \in \mathbf{R}^{k_i \times k_i}$, $i = 1, \dots, M$, $d_i \in \mathbf{R}$, $i = 1, \dots, N$, with $\sigma_{\max}(\Delta) \leq 1$. Then, Δ satisfies every IQC from

$$\mathbf{\Pi}^{\text{par}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ -Y(j\omega) & -X(j\omega) \end{bmatrix}, \\ X(j\omega) = X(j\omega)^* \text{ and } Y(j\omega)^* = -Y(j\omega) \text{ in } \mathcal{W}, \\ \text{for some } \epsilon > 0, X(j\omega) \geq 2\epsilon I, \text{ for all } \omega \in \mathbf{R} \end{array} \right. \right\},$$

where

$$\mathcal{W} = \left\{ \begin{array}{l} \text{diag}(w_1 I_{k_1}, \dots, w_M I_{k_M}, \\ W_1, \dots, W_N) \end{array} \left| \begin{array}{l} w_i \in \mathbf{C}, i = 1, \dots, M \\ W_i \in \mathbf{C}^{\ell_i \times \ell_i}, i = 1, \dots, N \end{array} \right. \right\}. \quad (14.4.31)$$

(See for example [50, 509].)

For such uncertainties, Theorem 14.4.2 can be immediately used to obtain sufficient condition that guarantees the stability of the closed loop system. Let \mathcal{P} denote the subset of real matrices that lie in \mathcal{W} , i.e., $\mathcal{P} = \mathcal{W} \cap \mathbf{R}^{m \times m}$. Then, a subset $\mathbf{\Pi}_{\text{fin}}^{\text{par}}$ of $\mathbf{\Pi}^{\text{par}}$, that is described by a finite number of variables, can be defined as follows. Let $W^{(1)}, \dots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Let

$$\mathbf{\Theta}^{\text{par}} = \left\{ \left[\begin{array}{cccc} \theta_{11} & \theta_{12} & \cdots & \theta_{1N} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N1} & \theta_{N2} & \cdots & \theta_{NN} \end{array} \right] \left| \theta_{ij} \in \mathcal{P} \right. \right\}. \quad (14.4.32)$$

Then, a subset $\mathbf{\Pi}_{\text{fin}}^{\text{par}}$ of $\mathbf{\Pi}^{\text{par}}$, that is described by a finite number of variables is given by (14.4.27), where (A_W, B_W, C_W) is any state space realization of $[W^{(1)}(s)^T \dots W^{(N-1)}(s)^T]^T$, $D_W = I$, and

$$\mathbf{T}^{\text{par}} = \left\{ \left[\begin{array}{cc} \Theta + \Theta^T & \Phi - \Phi^T \\ \Phi^T - \Phi & -(\Theta + \Theta^T) \end{array} \right] \middle| \Theta, \Phi \in \mathbf{\Theta}^{\text{par}} \right\}. \quad (14.4.33)$$

Note that the choice of $W^{(i)}$ is *ad hoc*, and the robust analysis result will certainly depend on this choice. However, it can be shown (see [147]) that the actual choice of the $W^{(i)}$ is immaterial, provided the set of $W^{(i)}$ s is chosen to be “rich enough”.

14.4.3 Structured dynamic uncertainties

Suppose that Δ is a linear time-invariant operator, such that for all $\omega \in \mathbf{R}$, $\Delta(j\omega) = \text{diag}(D_1, \dots, D_M, d_1 I_{\ell_1}, \dots, d_N I_{\ell_N})$, $D_i \in \mathbf{C}^{k_i \times k_i}$, $i = 1, \dots, M$, $d_i \in \mathbf{C}$, $i = 1, \dots, N$, with $\sigma_{\max}(\Delta(j\omega)) \leq 1$. Thus, Δ is a dynamic block-structured uncertainty, with an \mathbf{L}_2 -gain that does not exceed one.

Then, Δ satisfies every IQC from

$$\mathbf{\Pi}^{\text{LTI}} = \left\{ \Pi \middle| \begin{array}{l} \Pi(j\omega) = \left[\begin{array}{cc} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{array} \right], \quad X(j\omega) = X(j\omega)^* \in \mathcal{W}, \\ \text{for some } \epsilon > 0, \quad X(j\omega) \geq 2\epsilon I, \text{ for all } \omega \in \mathbf{R} \end{array} \right\},$$

where \mathcal{W} is defined in (14.4.31) (see for example [50, 509]). Similarly to the case of parametric uncertainty, $\mathbf{\Pi}_{\text{fin}}^{\text{LTI}} \subset \mathbf{\Pi}^{\text{LTI}}$ can be described by a finite number of variables. Let $W^{(1)}, \dots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Then, a subset $\mathbf{\Pi}_{\text{fin}}^{\text{LTI}}$ of $\mathbf{\Pi}^{\text{LTI}}$ described by a finite number of variables is given by (14.4.27), where (A_W, B_W, C_W) is any state space realization of $[W^{(1)}(s)^T \dots W^{(N-1)}(s)^T]^T$, $D_W = I$, and

$$\mathbf{T}^{\text{LTI}} = \left\{ \left[\begin{array}{cc} \Theta + \Theta^T & 0 \\ 0 & -(\Theta + \Theta^T) \end{array} \right] \middle| \Theta \in \mathbf{\Theta}^{\text{par}} \right\}. \quad (14.4.34)$$

14.5 STABILIZING CONTROLLER DESIGN FOR LFR SYSTEMS

Finally, we consider problem (P2) for system (14.2.4), i.e., the problem of synthesizing control laws that stabilize system (14.2.4). It is yet unknown how to gracefully extend the IQC-based robust stability analysis that we described in the previous section to synthesizing even the simplest of control laws, for instance state-feedback. We will therefore focus on the special case of an un-

certain system described by

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + B_p p(t) + B_u u(t), \\ q(t) &= C_q x(t) + D_{qp} p(t) + D_{qu} u(t), \\ y(t) &= C_y x(t) + D_{yp} p(t), \\ p(t) &= \Delta(t)q(t),\end{aligned}\tag{14.5.35}$$

where:

- The uncertainty $\Delta(t)$ is measurable in real-time.
- The “ L_2 gain” of $\Delta(t)$ does not exceed one (see (14.4.28)).

For this case, we will describe how state-feedback and gain-scheduled output feedback control laws can be designed using quadratic Lyapunov functions and SDP.

14.5.1 Quadratic stability analysis of LFR systems

For the robust stability analysis problem, we set the control input u to be identically zero. Then, the Lyapunov function $V(x(t)) = x(t)^T P x(t)$ with $P = P^T > 0$ decreases along the trajectories of (14.5.35) for all Δ if

$$x(t)^T (A^T P + P A) x(t) + 2x(t)^T P B_p p(t) < 0 \text{ for nonzero } x(t),$$

whenever

$$\int_0^T p(t)^T p(t) dt \leq \int_0^T q(t)^T q(t) dt.$$

It can be shown that this holds if

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}^T \begin{bmatrix} PA + A^T P & P B_p \\ B_p^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} < 0,$$

for every x and p that satisfy (14.5.36)

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}^T \begin{bmatrix} C_q^T C_q & C_q^T D_{qp} \\ D_{qp}^T C_q & D_{qp}^T D_{qp} - I \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \geq 0.$$

There exists P such that condition (14.5.36) holds if there exists P such that the following LMIs are feasible:

$$P = P^T > 0, \quad \begin{bmatrix} PA + A^T P + C_q^T C_q & P B_p + C_q^T D_{qp} \\ B_p^T P + D_{qp}^T C_q & D_{qp}^T D_{qp} - I \end{bmatrix} < 0. \tag{14.5.37}$$

Thus, we have an LMI condition that is sufficient for the quadratic stability of system (14.5.35).

14.5.2 State feedback controller design for LFR systems

Consider system (14.5.35) with constant state feedback $u(t) = Kx(t)$. The closed loop system is

$$\begin{aligned} \frac{d}{dt}x(t) &= (A + B_u K)x(t) + B_p p(t), \\ q(t) &= (C_q + D_{qu} K)x(t) + D_{qp} p(t), \\ p(t) &= \Delta(t)q(t), \end{aligned} \quad (14.5.38)$$

Using the sufficient condition for robust stability (14.5.37), and following a line of argument similar to that in the derivation of stabilizing state feedback laws for polytopic systems, we conclude that there exists a stabilizing constant state feedback controller for system (14.5.35) if there exist $Q = Q^T > 0$ and Y such that the following LMI holds.

$$\begin{bmatrix} AQ + QA^T + B_p B_p^T + B_u Y + Y^T B_u^T & B_p D_{qp}^T + QC_q^T + Y^T D_{qu}^T \\ D_{qp} B_p^T + C_q Q + D_{qu} Y & D_{qp} D_{qp}^T - I \end{bmatrix} < 0. \quad (14.5.39)$$

$V(\psi) = \psi^T Q^{-1} \psi$ is a Lyapunov function that proves the quadratic stability of the closed loop system (14.5.38), and $K = YQ^{-1}$ is the corresponding stabilizing state feedback gain.

14.5.3 Gain-scheduled output feedback controller design

As with polytopic systems, the general output feedback controller synthesis problem for LFR systems is not a convex feasibility problem. However, if the uncertainty $\Delta(t)$ can be measured in real time, the design of a gain-scheduled controller, i.e., one that depends on $\Delta(t)$, turns out to be an SDP problem.

The closed-loop system, with the gain-scheduled controller enclosed in dotted lines, is shown in Figure 14.3. The controller consists of an LTI system \mathcal{K}_{LTI} , with the uncertainty $\Delta(t)$ appearing in a feedback configuration. Let the state space representation of \mathcal{K}_{LTI} be

$$\begin{aligned} \frac{d}{dt}x_k(t) &= A_k x_k(t) + B_{k1} y(t) + B_{k2} v(t), \\ u(t) &= C_{k1} x_k(t) + D_{k11} y(t) + D_{k12} v(t), \\ f(t) &= C_{k2} x_k(t) + D_{k21} y(t) + D_{k22} v(t). \end{aligned} \quad (14.5.40)$$

Then, the equations governing the closed loop system in Figure 14.3 are

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} &= A_{\text{cl}} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} + B_{\text{cl}} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix}, \\ \begin{bmatrix} f(t) \\ q(t) \end{bmatrix} &= C_{\text{cl}} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} + D_{\text{cl}} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix}, \\ \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} &= \begin{bmatrix} \Delta(t) & \\ & \Delta(t) \end{bmatrix} \begin{bmatrix} f(t) \\ q(t) \end{bmatrix}, \end{aligned}$$

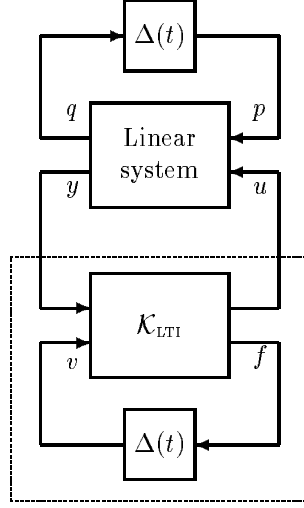


Figure 14.3 Gain scheduled output feedback control framework.

where

$$\begin{aligned} A_{\text{cl}} &= A_0 + \mathcal{B}\Omega\mathcal{C}, & B_{\text{cl}} &= B_0 + \mathcal{B}\Omega\mathcal{D}_{yp}, \\ C_{\text{cl}} &= C_0 + \mathcal{D}_{qu}\Omega\mathcal{C}, & D_{\text{cl}} &= D_0 + \mathcal{D}_{qu}\Omega\mathcal{D}_{yp}, \end{aligned}$$

with

$$\Omega = \begin{bmatrix} A_k & B_{k1} & B_{k2} \\ C_{k1} & D_{k11} & D_{k12} \\ C_{k2} & D_{k21} & D_{k22} \end{bmatrix} \quad (14.5.41)$$

and

$$\begin{aligned} A_0 &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, & B_0 &= \begin{bmatrix} 0 & B_p \\ 0 & 0 \end{bmatrix}, & \mathcal{C} &= \begin{bmatrix} 0 & I \\ C_y & 0 \\ 0 & 0 \end{bmatrix}, \\ C_0 &= \begin{bmatrix} 0 & 0 \\ C_q & 0 \end{bmatrix}, & D_0 &= \begin{bmatrix} 0 & 0 \\ 0 & D_{qp} \end{bmatrix}, & \mathcal{D}_{yp} &= \begin{bmatrix} 0 & 0 \\ D_{yp} & 0 \\ I & 0 \end{bmatrix}, \\ \mathcal{B} &= \begin{bmatrix} 0 & B_u & 0 \\ I & 0 & 0 \end{bmatrix}, & \mathcal{D}_{qu} &= \begin{bmatrix} 0 & 0 & I \\ 0 & D_{qu} & 0 \end{bmatrix}. \end{aligned} \quad (14.5.42)$$

The sufficient condition for robust stability (14.5.37) implies that there exists a stabilizing gain-scheduled output feedback controller for system (14.5.35) if there exist $P = P^T > 0$ and Ω such that

$$\begin{bmatrix} PA_{\text{cl}} + A_{\text{cl}}^T P + C_{\text{cl}}^T C_{\text{cl}} & PB_{\text{cl}} + C_{\text{cl}}^T D_{\text{cl}} \\ B_{\text{cl}}^T P + D_{\text{cl}}^T C_{\text{cl}} & D_{\text{cl}}^T D_{\text{cl}} - I \end{bmatrix} < 0, \quad (14.5.43)$$

Proceeding along the lines of the derivation of a gain-scheduled output feedback controller for polytopic systems, we conclude that there exists a full order stabilizing gain-scheduled controller such that condition (14.5.43) holds for some $P = P^T > 0$ if and only if there exist $R = R^T \in \mathbf{R}^{n \times n}$ and $S = S^T \in \mathbf{R}^{n \times n}$ such that the following LMIs hold:

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_q^T & B_p \\ C_q R & -I & D_{qp} \\ \hline B_p^T & D_{qp}^T & -I \end{array} \right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \quad (14.5.44)$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^T S + SA & SB_p & C_q^T \\ B_p^T S & -I & D_{qp}^T \\ \hline C_q & D_{qp} & -I \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \quad (14.5.45)$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0, \quad (14.5.46)$$

where N_R and N_S are matrices whose columns comprise the bases of the null spaces of $[B_u^T \ D_{qu}^T]$ and $[C_y \ D_{yp}]$ respectively.

Next, suppose there exist R and S such that the synthesis conditions are feasible. A gain-scheduled output feedback controller can then be designed as follows.

Step 1. Define $P_{12} = (S - R^{-1})^{1/2}$ and $Q_{12} = -RP_{12}$. It is easy to check that

$$P = \begin{bmatrix} S & P_{12} \\ P_{12}^T & I \end{bmatrix} > 0 \quad \text{and} \quad P^{-1} = \begin{bmatrix} R & Q_{12} \\ Q_{12}^T & I - P_{12}^T Q_{12} \end{bmatrix} > 0.$$

Step 2. Solve the following LMI for Ω .

$$X + U^T \Omega V + V^T \Omega^T U < 0, \quad (14.5.47)$$

where

$$X = \begin{bmatrix} A_0^T P + P A_0 & P B_0 & C_0^T \\ B_0^T P & -I & D_0^T \\ C_0 & D_0 & -I \end{bmatrix},$$

$$U = [\mathcal{B}^T P \quad 0 \quad \mathcal{D}_{qu}^T] \quad \text{and} \quad V = [\mathcal{C} \quad \mathcal{D}_{yp} \quad 0].$$

The feasible solution Ω in (14.5.47) comprises the state space matrices of the LTI part \mathcal{K}_{LTI} of a stabilizing gain-scheduled output feedback controller (see (14.5.40)).

14.6 CONCLUSION

We have described some examples of the application of SDP in robust control, specifically the problems of robust stability analysis and stabilizing controller design of system models that incorporate model uncertainties. The numerical solution of engineering problems via an SDP reformulation continues to be a popular solution technique not only in robust control, but also in other related areas of system theory, such as communications and signal processing.